

On the Deligne-Simpson problem*

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To prof. V.I. Arnold

Abstract

The Deligne-Simpson problem is formulated like this: *give necessary and sufficient conditions for the choice of the conjugacy classes $C_j \subset SL(n, \mathbf{C})$ or $c_j \subset sl(n, \mathbf{C})$ so that there exist irreducible $(p+1)$ -tuples of matrices $M_j \in C_j$ or $A_j \in c_j$ satisfying the equality $M_1 \dots M_{p+1} = I$ or $A_1 + \dots + A_{p+1} = 0$.*

We solve the problem for generic eigenvalues with the exception of the case of matrices M_j when the greatest common divisor of the numbers $\Sigma_{j,l}(\sigma)$ of Jordan blocks of a given matrix M_j , with a given eigenvalue σ and of a given size l (taken over all j, σ, l) is > 1 . Generic eigenvalues are defined by explicit algebraic inequalities. For such eigenvalues there exist no reducible $(p+1)$ -tuples.

The matrices M_j and A_j are interpreted as monodromy operators of regular linear systems and as matrices-residua of fuchsian ones on Riemann's sphere.

Key words: generic eigenvalues, (poly)multiplicity vector, corresponding Jordan normal forms, monodromy operator.

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1 Introduction

1.1 Formulation of the problem

The problem which is dealt with in the present paper is connected with the theory of *fuchsian linear systems* on Riemann's sphere, i.e. meromorphic linear systems of differential equations with logarithmic poles. Such a system can be represented as follows:

$$\dot{X} = \left(\sum_{j=1}^{p+1} \frac{A_j}{t - a_j} \right) X \quad (1)$$

where "·" denotes d/dt , $t \in \mathbf{CP}^1$, $A_j \in gl(n, \mathbf{C})$, the $p+1$ poles a_j are distinct and we assume without restriction that there is no pole at infinity. This last condition implies that the sum of the *matrices-residua* A_j is 0:

$$\sum_{j=1}^{p+1} A_j = 0 \quad (2)$$

Fuchsian systems are a particular case of *regular* systems, i.e. linear systems whose solutions when restricted to sectors centered at the poles have a moderate growth rate when the argument tends to the pole: $\|X\| = O(|t - a_j|^{N_j})$ for some $N_j \in \mathbf{R}$.

It is more convenient to consider the dependent variables X as an $n \times n$ -matrix, i.e. to consider simultaneously n linearly independent vector-solutions. This is what we do.

Fix a base point a_0 different from the poles a_j . Fix the value $B \in GL(n, \mathbf{C})$ of the solution for $t = a_0$. For each pole a_j define a closed contour Γ_j containing a_0 and freely homotopic to a positive loop around a_j . The contour Γ_j consists of a line segment $[a_0, x_j]$ where x_j is close to a_j , of the circumference Θ_j centered at a_j , passing through x_j and circumventing a_j counterclockwise (we choose x_j so close to a_j that no other pole of the system lies inside or on Θ_j), and of the line segment $[x_j, a_0]$. We assume that for $i \neq j$ one has $\Gamma_i \cap \Gamma_j = \{a_0\}$ and that when one turns around a_0 clockwise the indices of the contours change from 1 to $p+1$.

The value at a_0 of the analytic continuation of the solution along Γ_j is representable in the form BM_j . The matrix $M_j \in GL(n, \mathbf{C})$ is by definition the matrix of the *monodromy operator* corresponding to the class of homotopy equivalence of the contour Γ_j . For a choice of contours Γ_j like above one has

$$M_1 \dots M_{p+1} = I \quad (3)$$

which is the multiplicative analog of (2). The monodromy operators generate the *monodromy group* which is invariant under linear transformations of the dependent variables meromorphically depending on the time (and, up to conjugacy, the only such invariant).

Remark 1 *Note that with this definition the monodromy group is an antirepresentation of $\pi_1(\mathbf{CP}^1 \setminus \{a_1, \dots, a_{p+1}\})$ into $GL(n, \mathbf{C})$ (because to the product of contours $\Gamma_i \Gamma_j$ there corresponds the monodromy operator $M_j M_i$; product in the sense of concatenation). To obtain a representation one has to consider the matrices M_j^{-1} . In the paper we refer to the $(p+1)$ -tuples of matrices also as to representations.*

The *Deligne-Simpson problem (DSP)* is formulated as follows:

For what $(p+1)$ -tuples of conjugacy classes $C_j \subset SL(n, \mathbf{C})$ do there exist irreducible $(p+1)$ -tuples of matrices $M_j \in C_j$ satisfying (3) ? (multiplicative version).

For what $(p+1)$ -tuples of conjugacy classes $c_j \subset sl(n, \mathbf{C})$ do there exist irreducible $(p+1)$ -tuples of matrices $A_j \in c_j$ satisfying (2) ? (additive version).

We give the basic result in Subsection 1.5 (see Theorems 17, 19 and 20) followed by a plan of the paper, after introducing some definitions in the next three subsections.

Remarks 2 1) "Irreducible" means "not having a common proper invariant subspace"; in other words, impossible to bring the $(p+1)$ -tuple to a block upper-triangular form by simultaneous conjugation. The problem could be formulated without the requirement of irreducibility and it would be another problem which we do not consider here. However, we consider the problem with "irreducible" replaced by "with trivial centralizer", see Theorem 20.

2) In the multiplicative version (i.e. for matrices M_j) the problem was formulated by P.Deligne, and C.Simpson was the first to obtain results towards its solution, see [Si1] and [Si2]. Simpson's result is cited in Remarks 18.

3) The case of nilpotent matrices A_j and of unipotent matrices M_j was considered by the author in [Ko1] and [Ko2]. One of the results from [Ko2] is used in the present paper, see Theorem 13.

4) We treat the two versions of the problem (additive and multiplicative) in parallel. The multiplicative should be considered as more important because the monodromy group is a meromorphic invariant (up to conjugacy) whereas the $(p+1)$ -tuple of matrices-residua is not.

5) One can replace $sl(n, \mathbf{C})$ or $SL(n, \mathbf{C})$ by $gl(n, \mathbf{C})$ or $GL(n, \mathbf{C})$; this is what we do when we solve the problem because in the process of solving it one encounters matrices A_j and M_j not from $sl(n, \mathbf{C})$ (resp. not from $SL(n, \mathbf{C})$).

6) Notice that the Deligne-Simpson problem is formulated in a purely algebraic way, without reference to fuchsian or regular systems. Yet they explain the interest in solving it.

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1.2 Generic eigenvalues

Definition 3 Call Jordan normal form of size n a family $J^n = \{b_{i,l}\}$ ($i \in I_l$, $I_l = \{1, \dots, s_l\}$, $l \in L$) of positive integers $b_{i,l}$ whose sum is n . The set L is the one of indices of eigenvalues (all distinct) and I_l is the set of indices of Jordan blocks with the l -th eigenvalue; $b_{i,l}$ is the size of the i -th block with this eigenvalue. An $n \times n$ -matrix Y has the Jordan normal form J^n (notation: $J(Y) = J^n$) if its eigenvalues can be indexed by L in such a way that to its distinct eigenvalues δ_l ($l \in L$) there belong Jordan blocks of sizes $b_{i,l}$. We assume that for every fixed l one has $b_{1,l} \geq b_{2,l} \geq \dots \geq b_{s_l,l}$.

Remark 4 The basic result of this paper depends actually not on the conjugacy classes but only on the Jordan normal forms of the matrices A_j or M_j provided that the eigenvalues remain generic, see the definition of generic eigenvalues below.

We assume that the following necessary conditions for existence of irreducible $(p+1)$ -tuples of matrices A_j or M_j (satisfying (2) or (3)) hold:

$$\sum_{j=1}^{p+1} \text{Tr} C_j = 0 \quad , \quad \prod_{j=1}^{p+1} \det C_j = 1 \quad (4)$$

Denote by $\lambda_{k,j}$, $\sigma_{k,j}$ the eigenvalues of A_j , M_j (they are *not* presumed distinct; a *multiplicity* of an eigenvalue is by definition the number of eigenvalues equal to it including the eigenvalue itself). When A_j are the matrices-residua of a fuchsian system with monodromy operators M_j , then one has $\sigma_{k,j} = \exp(2\pi i \lambda_{k,j})$. Equation (4) admits the following equivalent form:

$$\sum_{k=1}^n \sum_{j=1}^{p+1} \lambda_{k,j} = 0 \quad , \quad \prod_{k=1}^n \prod_{j=1}^{p+1} \sigma_{k,j} = 1 \quad (5)$$

(see (2) and (3)). From now on we presume that the eigenvalues satisfy conditions (5). Call *non-genericity relation* any equality of the form

$$\sum_{j=1}^{p+1} \sum_{k \in \Phi_j} \lambda_{k,j} = 0 \quad , \quad \text{resp.} \quad \prod_{j=1}^{p+1} \prod_{k \in \Phi_j} \sigma_{k,j} = 1 \quad (\gamma)$$

where the sets Φ_j contain one and the same number s of indices (with $1 < s < n$) for all j . Eigenvalues satisfying none of these relations are called *generic*.

Eigenvalues $\lambda_{k,j}$ satisfying none of the relations (γ) modulo \mathbf{Z} are called *strongly generic*. If the eigenvalues $\lambda_{k,j}$ are strongly generic, then the eigenvalues $\sigma_{k,j}$ are generic.

Remarks 5 1) *Reducible (i.e. block upper-triangular up to conjugacy) $(p+1)$ -tuples of matrices exist only for non-generic eigenvalues and if the $(p+1)$ -tuple is in block upper-triangular form, then the eigenvalues of its restriction to each diagonal block satisfy some relation (γ) .*

2) *If in (γ) one replaces each of the sets Φ_j by its complement in $\{1, \dots, n\}$, then one obtains an equivalent non-genericity relation.*

1.3 The quantities r_j and d_j

For a conjugacy class C in $GL(n, \mathbf{C})$ or $gl(n, \mathbf{C})$ denote by $d(C)$ its dimension and for a matrix Y from C set $r(C) := \min_{\lambda \in \mathbf{C}} \text{rank}(Y - \lambda I)$. The integer $n - r(C)$ is the maximal number of Jordan blocks of $J(Y)$ with one and the same eigenvalue. Set $d_j := d(C_j)$ (resp. $d(c_j)$), $r_j := r(C_j)$ (resp. $r(c_j)$). The quantities $r(C)$ and $d(C)$ depend only on the Jordan normal form $J(Y) = J^n$, so we write sometimes $r(J^n)$ and $d(J^n)$.

The following proposition was proved in the multiplicative version by C.Simpson in [Sil]. We give a proof for both versions here:

Proposition 6 *A necessary condition for the existence of irreducible $(p+1)$ -tuples satisfying respectively (2) or (3) is the following couple of inequalities to hold:*

$$\begin{aligned} d_1 + \dots + d_{p+1} &\geq 2n^2 - 2 & (\alpha_n) \\ \text{for all } j \ (r_1 + \dots + r_j + \dots + r_{p+1}) &\geq n & (\beta_n) \end{aligned}$$

Condition (β_n) is generalized in Proposition 9. Both propositions are proved in Section 12.

Remark 7 *Neither of the inequalities (α_n) and (β_n) follows from the other one. If $n \geq 4$ is even, $p = 2$, the matrices A_j are diagonalizable and the multiplicity vectors of the eigenvalues of A_1, A_2, A_3 equal respectively $(1, \dots, 1)$, $(n/2, n/2)$, $(n/2, n/2)$, then (β_n) holds while (α_n) does not. If n is even, $n + 2 = 2p$, $p \geq 3$ and the multiplicity vectors of the eigenvalues of the diagonalizable matrices A_j equal $(1, \dots, 1)$, $(n-1, 1)$, \dots , $(n-1, 1)$, then (α_n) holds while (β_n) does not.*

Definition 8 *In the additive version we say that the DSP is solvable (resp. weakly solvable) for given Jordan normal forms J_j^n and for given eigenvalues if there exists an irreducible $(p+1)$ -tuple of matrices A_j satisfying condition (2) with $J(A_j) = J_j^n$ and with the given eigenvalues (resp. if there exists such a $(p+1)$ -tuple of matrices A_j with a trivial centralizer). In the multiplicative version one replaces in the definition the matrices A_j by matrices M_j satisfying condition (3).*

Proposition 9 1) For prescribed conjugacy classes c_j of the matrices A_j (not necessarily diagonalizable or with generic eigenvalues) denote by $r_j(b)$ the rank of the matrix $A_j - bI$, $b \in \mathbf{C}$, $A_j \in c_j$. A necessary condition for the solvability of the DSP for matrices $A_j \in c_j$ is the following inequality:

$$\min_{b_i \in \mathbf{C}, b_1 + \dots + b_{p+1} = 0} (r_1(b_1) + \dots + r_{p+1}(b_{p+1})) \geq 2n$$

2) A necessary condition for the solvability of the DSP for matrices M_j from prescribed conjugacy classes C_j (not necessarily diagonalizable or with generic eigenvalues) is the following inequality:

$$\min_{b_j \in \mathbf{C}^*, b_1 \dots b_{p+1} = 1} (\text{rk}(b_1 M_1 - I) + \dots + \text{rk}(b_{p+1} M_{p+1} - I)) \geq 2n$$

Remarks 10 1) Remind that the $(p+1)$ -tuples (A_1, \dots, A_{p+1}) and $(A_1 - b_1 I, \dots, A_{p+1} - b_{p+1} I)$, $b_1 + \dots + b_{p+1} = 0$ are simultaneously irreducible. The above minimum is obtained for a $(p+1)$ -tuple (b_1, \dots, b_{p+1}) in which at least p of the numbers b_j are eigenvalues of the corresponding matrices A_j . A similar remark is true for the matrices M_j as well.

2) The proposition generalizes condition (β_n) . In the case of generic eigenvalues it coincides with it, in the case of non-generic ones it implies it and is stronger than it.

For the formulation of the basic result it will be essential whether the quantities r_j satisfy the inequality

$$(r_1 + \dots + r_{p+1}) \geq 2n \quad (\omega_n)$$

Remarks 11 1) Evidently, condition (ω_n) implies condition (β_n) (because $r_j < n$), but condition (ω_n) is not necessary for the existence of irreducible $(p+1)$ -tuples (e.g. for $p = 2$, $n = 2$ there exist irreducible triples of matrices each with two distinct eigenvalues, i.e. $r_1 = r_2 = r_3 = 1$ and (ω_2) does not hold).

2) Condition (ω_n) arises when the Deligne-Simpson problem is considered for nilpotent matrices A_j and for unipotent matrices M_j . For such matrices the eigenvalues are “mostly non-generic”, i.e. they satisfy all possible non-genericity relations. Moreover, for such matrices condition (ω_n) is necessary for the existence of irreducible $(p+1)$ -tuples – it coincides with the necessary conditions from Proposition 9. It turns out that condition (ω_n) is almost sufficient as well in the following sense.

Definition 12 A $(p+1)$ -tuple of matrices A_j whose sum is 0 or of matrices M_j whose product is I is said to define a nice representation (or for short, to be nice) if its centralizer is trivial and either the $(p+1)$ -tuple is irreducible or it is reducible and one can conjugate it to a block upper-triangular form in which the diagonal blocks are all of sizes $g_i > 1$ and define irreducible representations. Thus the matrix algebra \mathcal{A} defined by the matrices A_j or $M_j - I$ contains a non-degenerate matrix (i.e. with non-zero determinant) – by the Burnside theorem the restrictions to the diagonal blocks of \mathcal{A} equal $gl(g_i, \mathbf{C})$.

Theorem 13 *If for the nilpotent (resp. unipotent) conjugacy classes c_j (resp. C_j) condition (ω_n) holds and the following four particular cases are avoided, then there exist nice $(p+1)$ -tuples of matrices $A_j \in c_j$ whose sum is 0 (resp. of matrices $M_j \in C_j$ whose product is I). In the four particular cases each conjugacy class has Jordan blocks of one and the same size (denoted by l_j). The cases are:*

- 1) $n = 2k$, $k > 1$, $p = 3$, $l_1 = l_2 = l_3 = l_4 = 2$;
- 2) $n = 3k$, $k > 1$, $p = 2$, $l_1 = l_2 = l_3 = 3$;
- 3) $n = 4k$, $k > 1$, $p = 2$, $l_1 = l_2 = 4$, $l_3 = 2$;
- 4) $n = 6k$, $k > 1$, $p = 2$, $l_1 = 6$, $l_2 = 3$, $l_3 = 2$.

The above theorem is part of Theorem 34 from [Ko2]. If another four particular cases are avoided, then there exist $(p+1)$ -tuples defining irreducible representations, see [Ko2].

1.4 (Poly)multiplicity vectors

Definition 14 *A polymultiplicity vector (PMV) is by definition a $(p+1)$ -tuple of multiplicity vectors (MVs), i.e. vectors whose components are non-negative integers, their sum (called the length of the MV) being equal to n . The MVs and PMVs with which we deal in this paper are defined by the multiplicities of the eigenvalues of the matrices A_j or M_j . (We allow zero components for the sake of convenience.) Call a PMV simple (resp. non-simple) if the greatest common divisor of all its non-zero components equals 1 (resp. if not).*

Remark 15 *In the case of matrices A_j generic eigenvalues exist only for simple PMVs. Indeed, if all multiplicities are divisible by $1 < q \in \mathbf{Z}$, then the sum of all eigenvalues with multiplicities divided by q equals 0 which is a non-genericity relation (γ) . In the case of matrices M_j the divisibility by q would imply only that the product of all eigenvalues with multiplicities divided by q equals one of the roots of unity of q -th order, not necessarily 1, i.e. a non-genericity relation might or might not hold. However, in both cases there exist generic eigenvalues for every simple PMV. Generic eigenvalues form a Zariski open dense subset in the set of all eigenvalues with a fixed simple PMV. The latter set is a linear space in the additive version and a non-singular variety in the multiplicative one, see condition (5).*

For a PMV of length n we use the notation $\Lambda^n = (\Lambda_1^n, \dots, \Lambda_{p+1}^n)$ where Λ_j^n are the MVs. For diagonalizable matrices the MV Λ_j^n to have only one component implies A_j or M_j to be scalar.

Set $\Lambda_j^n = (m_{1,j}, \dots, m_{k_j,j})$, $m_j = \max_i m_{i,j}$. For a diagonalizable conjugacy class one has $r_j = n - m_j$, $d_j = n^2 - \sum_{i=1}^{k_j} (m_{i,j})^2$. In accordance with the corresponding definitions for Jordan normal forms, we say that Λ^n satisfies Condition (α_n) (Condition (β_n) , Condition (ω_n)) if $d_1 + \dots + d_{p+1} \geq 2n^2 - 2$ (if $\min_j (r_1 + \dots + r_{p+1} - r_j) \geq n$, if $r_1 + \dots + r_{p+1} \geq 2n$).

1.5 Formulation of the basic result

For a given $(p+1)$ -tuple $(J_1^n, \dots, J_{p+1}^n)$ of Jordan normal forms with $n > 1$ (the upper index indicates the size of the matrices), which satisfies condition (β_n) and does not satisfy condition (ω_n) set $n_1 = r_1 + \dots + r_{p+1} - n$. Hence, $n_1 < n$ and $n - n_1 \leq n - r_j$ for all j . Define the $(p+1)$ -tuple of Jordan normal forms $J_j^{n_1}$ as follows: to obtain the Jordan normal form $J_j^{n_1}$ from J_j^n one chooses one of the eigenvalues of J_j^n with greatest number $n - r_j$ of Jordan blocks, then decreases by 1 the sizes of the $n - n_1$ smallest Jordan blocks with this eigenvalue and deletes the Jordan blocks of size 0. We write this symbolically in the form

$$\Psi : (J_1^n, \dots, J_{p+1}^n) \mapsto (J_1^{n_1}, \dots, J_{p+1}^{n_1})$$

For a given $(p+1)$ -tuple of Jordan normal forms $J^n = (J_1^n, \dots, J_{p+1}^n)$ define a sequence of $(p+1)$ -tuples of Jordan normal forms J^{n_ν} , $\nu = 0, \dots, s$ by iterating the map Ψ as long as it is defined; we set $n_0 = n$.

Remark 16 Notice that $n > n_1 > \dots > n_s$ (we define n_1 only if J^n does not satisfy condition (ω_n) , hence, $n_1 < n$ etc.).

Theorem 17 For given Jordan normal forms J_j^n with a simple PMV and for generic eigenvalues the DSP is solvable for matrices A_j or M_j if and only if the following two conditions hold:

- i) The $(p+1)$ -tuple of Jordan normal forms J_j^n satisfies the inequality (β_n) ;
- ii) Either the $(p+1)$ -tuple of Jordan normal forms $J_j^{n_s}$ satisfies the inequality (ω_{n_s}) or $n_s = 1$.

Remarks 18 1) The theorem holds whichever choice of eigenvalue with maximal number of Jordan blocks is made to define Ψ (one choice is sufficient, it holds automatically for all other choices; the numbers n_k are the same for all choices). See Remarks 56.

2) Condition (α_n) does not appear explicitly in the formulation of the theorem. However, it is implicitly present because if the $(p+1)$ -tuple of Jordan normal forms $J_j^{n_\nu}$ satisfies condition (α_{n_ν}) , then the $(p+1)$ -tuple of Jordan normal forms $J_j^{n_{\nu+1}}$ satisfies condition $(\alpha_{n_{\nu+1}})$, see Corollary 61, hence, it suffices to check that condition (α_{n_s}) holds for the $(p+1)$ -tuple of Jordan normal forms $J_j^{n_s}$. This is true – if $n_s = 1$, then $d(J_j^{n_s}) = 0$ for all j and (α_{n_s}) is an equality. If there holds (ω_{n_s}) , then there holds (α_{n_s}) as well and it is a strict inequality, see Remark 62.

3) In [Si1] C.Simpson proved in the multiplicative version of the problem that for generic eigenvalues and one of the matrices M_j having distinct eigenvalues the necessary and sufficient conditions for the solvability of the DSP is the inequalities (α_n) and (β_n) to hold. The author has shown in [Ko3] that this is true also if one of the matrices M_j or A_j has only eigenvalues of multiplicity ≤ 2 . With Theorem 17 one gets rid of the condition on the multiplicities of the eigenvalues of one of the matrices.

4) The case when condition (α_n) is an equality for matrices M_j is considered in detail in [Ka] where it is explained how to construct such irreducible $(p+1)$ -tuples (called rigid) of matrices M_j . Examples of existence of rigid $(p+1)$ -tuples can be found in [Si1], [Gl] and [Ko3].

The theorem does not cover the case of matrices M_j , when the PMV of the eigenvalues $\sigma_{k,j}$ is non-simple but the eigenvalues are generic. In this case the following theorem clarifies partially the situation.

Denote by $\Sigma_{j,l}(\sigma)$ the number of Jordan blocks of M_j of size l , with eigenvalue σ , and by d the greatest common divisor of the numbers $\Sigma_{j,l}(\sigma)$ (over all j and l , over all eigenvalues σ). Even for non-simple PMV one has

Theorem 19 If $d = 1$, then for generic eigenvalues the necessary and sufficient condition for the solvability of the DSP for matrices M_j with given Jordan normal forms J_j^n is the conditions i) and ii) from Theorem 17 to hold.

In the case when the eigenvalues are not necessarily generic there holds the following

Theorem 20 1) If $d = 1$ and if inequality (α_n) is strict, then conditions i) and ii) from Theorem 17 are necessary and sufficient for the weak solvability of the DSP in the case of matrices A_j for any eigenvalues.

2) If $d = 1$ and if inequality (α_n) is strict, then conditions i) and ii) from Theorem 17 are necessary and sufficient for the solvability of the DSP in the case of matrices M_j with generic eigenvalues.

For matrices A_j part 1) of the theorem is not true if (α_n) is an equality. Example: $p = 2$, $n = 2$ and each matrix A_j is nilpotent, of rank 1. Such Jordan normal forms satisfy conditions i) and ii) from Theorem 17, but the triple is (up to conjugacy) upper-triangular and its centralizer is generated by I and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

1.6 Plan of the paper

The next three sections introduce the basic ingredients used to prove Theorems 17, 19 and 20. In Section 2 we describe *the basic technical tool* which is a way to obtain irreducible $(p+1)$ -tuples of matrices by deforming $(p+1)$ -tuples of matrices with trivial centralizers. Such a deformation allows one to keep the Jordan normal forms of the $p+1$ matrices the same while changing the eigenvalues. It allows also to change their Jordan normal forms to new ones.

In Section 3 we introduce a result due to A.H.M. Levelt describing the structure of the solution to a regular system in a neighbourhood of a pole. Lemma 33 from that section is important because it is used further to transform solving the DSP in the multiplicative version into solving it in the additive one. This lemma gives also a hint why the answers to the DSP in both versions are the same in the cases covered by this paper (see Corollary 34). (We should note that there are cases not covered by the present paper in which the formulation of the result in the multiplicative version is more complicated than the one in the additive version, see Remark 15.)

Before proving Theorem 17 we prove its weakened version first:

Theorem 21 For given Jordan normal forms whose PMV is simple conditions i) and ii) from Theorem 17 are necessary and sufficient for the solvability of the DSP (for matrices A_j or M_j) for all eigenvalues from some Zariski open dense subset in the set of all generic eigenvalues with this PMV.

In Section 4 we explain how to reduce the proof of Theorem 21 to the case of diagonalizable matrices A_j or M_j . This reduction uses the basic technical tool.

In Section 5 we formulate the result (Theorem 58) in the case of diagonalizable matrices A_j or M_j . We deduce Theorem 21 from Theorem 58 at the end of that section.

In Section 6 we prove the sufficiency and in Section 7 we prove the necessity of conditions i) and ii) of Theorem 17 for the existence of irreducible $(p+1)$ -tuples of diagonalizable matrices. This is the proof of Theorem 58. In principle, when inequality (α_n) is strict, the sufficiency follows from Theorem 20. We prove the sufficiency in Section 6 to cover also the case when (α_n) is an equality.

In the case when (α_n) is a strict inequality Theorem 17 follows from Theorem 21 and from Theorem 20. The latter in the case of generic eigenvalues provides the existence of irreducible $(p+1)$ -tuples of matrices.

In the case when (α_n) is an equality Theorem 17 results from

Theorem 22 *For given Jordan normal forms for which (α_n) is an equality, conditions i) and ii) from Theorem 17 are necessary and sufficient for the solvability of the DSP (for matrices A_j or M_j) for any generic eigenvalues.*

The theorem is proved in Section 10. In the proof we use Theorem 21. In fact, the sufficiency and the necessity being already proved respectively in Sections 6 and 7 there remains only to be proved in Section 10 that if conditions i) and ii) from Theorem 17 hold, then for such Jordan normal forms (admitting generic eigenvalues) the DSP is solvable for *all* generic eigenvalues.

Theorem 20 is proved in Section 9 after some preparation, i.e. after Section 8 where we discuss adjacency of nilpotent orbits.

In Section 11 we prove Theorem 19. In the proof we use Theorem 21.

Section 12 contains the proofs of Propositions 6 and 9.

2 The basic technical tool

2.1 The basic technical tool in the additive version

Definition 23 *Call basic technical tool the procedure described below. One starts with a $(p+1)$ -tuple of matrices A_j (not necessarily irreducible) satisfying (2) and having a trivial centralizer. Set $A_j = Q_j^{-1}G_jQ_j$, G_j being Jordan matrices. One looks for a $(p+1)$ -tuple of matrices \tilde{A}_j of the form*

$$\tilde{A}_j = (I + \sum_{i=1}^l \varepsilon_i X_{j,i}(\varepsilon))^{-1} Q_j^{-1} (G_j + \sum_{i=1}^l \varepsilon_i V_{j,i}(\varepsilon)) Q_j (I + \sum_{i=1}^l \varepsilon_i X_{j,i}(\varepsilon))$$

where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_l) \in (\mathbf{C}^l, 0)$ and $V_{j,i}(\varepsilon)$ are given matrices analytic in ε (in each concrete application their properties will be specified). One has $\text{tr}(\sum_{j=1}^{p+1} \sum_{i=1}^l \varepsilon_i V_{j,i}(\varepsilon)) \equiv 0$. Often but not always one chooses $l = 1$ and $V_{j,1}$ such that the eigenvalues of the $(p+1)$ -tuple of matrices \tilde{A}_j are generic for $\varepsilon \neq 0$. One looks for X_j analytic in ε such that $\sum_{j=1}^{p+1} \tilde{A}_j \equiv 0$.

The condition $\sum_{j=1}^{p+1} \tilde{A}_j = 0$ yields (in first approximation w.r.t. ε)

$$\text{for all } i = 1, \dots, l \text{ one has } \sum_{j=1}^{p+1} (Q_j^{-1} V_{j,i}(0) Q_j + [A_j, X_{j,i}(0)]) = 0 \quad (6)$$

Proposition 24 *The centralizer of the p -tuple of matrices A_j ($j = 1, \dots, p$) is trivial if and only if the mapping $(sl(n, \mathbf{C}))^p \rightarrow sl(n, \mathbf{C})$, $(X_1, \dots, X_p) \mapsto \sum_{j=1}^p [A_j, X_j]$ is surjective.*

Proof: The mapping is not surjective if and only if the image of each mapping $X_j \mapsto [A_j, X_j]$ belongs to one and the same proper linear subspace of $sl(n, \mathbf{C})$. This means that there exists a matrix $0 \neq D \in sl(n, \mathbf{C})$ such that $\text{tr}(D[A_j, X_j]) = 0$ for all $X_j \in sl(n, \mathbf{C})$ and for $j = 1, \dots, p$. This is equivalent to $\text{tr}([D, A_j]X_j) = 0$ for all $X_j \in sl(n, \mathbf{C})$, i.e. $[D, A_j] = 0$ for $j = 1, \dots, p$.

The proposition is proved. \square

By Proposition 24, equation (6) is solvable w.r.t. $X_{j,i}(0)$. Hence, the equation $\sum_{j=1}^{p+1} \tilde{A}_j = 0$ is solvable w.r.t. $X_{j,i}$ for ε small enough by the implicit function theorem (we use the surjectivity here). If for $\varepsilon \neq 0$ small enough the eigenvalues of the matrices \tilde{A}_j are generic, then their $(p+1)$ -tuple is irreducible.

2.2 The basic technical tool in the multiplicative version

We explain here how the *basic technical tool* works in the multiplicative version. Given a $(p+1)$ -tuple of matrices M_j^1 with a trivial centralizer and satisfying condition (3), look for M_j of the form

$$M_j = (I + \sum_{i=1}^l \varepsilon_i X_{j,i}(\varepsilon))^{-1} (M_j^1 + \sum_{i=1}^l \varepsilon_i N_{j,i}(\varepsilon)) (I + \sum_{i=1}^l \varepsilon_i X_{j,i}(\varepsilon))$$

where the given matrices $N_{j,i}$ depend analytically on $\varepsilon \in (\mathbf{C}^l, 0)$ and one looks for matrices $X_{j,i}$ analytic in ε . (Like in the additive version one can set $M_j^1 = Q_j^{-1} G_j Q_j$, $N_{j,i} = Q_j^{-1} V_{j,i} Q_j$.)

The matrices M_j must satisfy equality (3). In first approximation w.r.t. ε this implies that

$$\text{for all } i = 1, \dots, l \text{ one has } \sum_{j=1}^{p+1} M_1^1 \dots M_{j-1}^1 ([M_j^1, X_{j,i}(0)] + N_{j,i}(0)) M_{j+1}^1 \dots M_{p+1}^1 = 0$$

or

$$\text{for all } i = 1, \dots, l \text{ one has } \sum_{j=1}^{p+1} P_{j-1} ([M_j^1, X_{j,i}(0)(M_j^1)^{-1}] + N_{j,i}(0)(M_j^1)^{-1}) P_{j-1}^{-1} = 0 \quad (7)$$

with $P_j = M_1^1 \dots M_j^1$, $P_{-1} = I$ (recall that there holds (3), therefore $M_j^1 M_{j+1}^1 \dots M_{p+1}^1 = P_{j-1}^{-1}$).

Equality (3) implies that $\det M_1 \dots \det M_{p+1} = 1$. One has $\det M_j = \det M_j^1 \det(I + \sum_{i=1}^l \varepsilon_i (M_j^1)^{-1} N_{j,i}) = (\det M_j^1) (1 + \sum_{i=1}^l \varepsilon_i \text{tr}((M_j^1)^{-1} N_{j,i}(0)) + \text{terms of order } \geq 2 \text{ in } \varepsilon)$. As $\det M_1^1 \dots \det M_{p+1}^1 = 1$, one has for all i $\text{tr}(\sum_{j=1}^{p+1} (M_j^1)^{-1} N_{j,i}(0)) = 0$ (terms of first order w.r.t. ε in $\det M_1 \dots \det M_{p+1}$).

Equation (7) can be written in the form

$$\text{for all } i = 1, \dots, l \text{ one has } \sum_{j=1}^{p+1} ([S_j, Z_{j,i}] + T_{j,i}) = 0 \quad (8)$$

with $S_j = P_{j-1} M_j^1 P_{j-1}^{-1}$, $Z_{j,i} = P_{j-1} X_{j,i}(0) (M_j^1)^{-1} P_{j-1}^{-1}$, $T_{j,i} = P_{j-1} N_{j,i}(0) (M_j^1)^{-1} P_{j-1}^{-1}$. The centralizers of the $(p+1)$ -tuples of matrices M_j^1 and S_j are the same (to be checked directly), i.e. they are both trivial. Hence, for all i the mappings

$$(sl(n, \mathbf{C}))^{p+1} \rightarrow sl(n, \mathbf{C}), (Z_{1,i}, \dots, Z_{p+1,i}) \mapsto \sum_{j=1}^{p+1} [S_j, Z_{j,i}]$$

are surjective (Proposition 24). Recall that for all i one has $\text{tr}(\sum_{j=1}^{p+1} (M_j^1)^{-1} N_{j,i}(0)) = 0$, i.e. $\text{tr}(\sum_{j=1}^{p+1} T_{j,i}) = 0$. Hence, equation (8) can be solved w.r.t. the unknown matrices $Z_{j,i}$ and, hence, equation (7) can be solved w.r.t. the matrices $X_{j,i}(0)$. By the implicit function theorem (we use the surjectivity here), one can find $X_{j,i}$ analytic in $\varepsilon \in (\mathbf{C}^l, 0)$, i.e. one can find the necessary matrices M_j .

A first application of the basic technical tool is the following

Lemma 25 *For a given $(p+1)$ -tuple of Jordan normal forms admitting generic eigenvalues denote by L (resp. by L') the set of all possible eigenvalues (resp. of all possible generic eigenvalues). If there exists a $(p+1)$ -tuple of matrices A_j (or M_j) with a trivial centralizer for some*

$\lambda_0 \in L$, then there exist $(p+1)$ -tuples of matrices A_j (or M_j) with trivial centralizers for all $\lambda \in L$ sufficiently close to λ_0 and for all λ from some Zariski open and dense subset of the connected component L'_0 of L' containing λ_0 .

Remark 26 In the multiplicative version of the DSP the set L can consist of several (namely, q) connected components if $q > 1$, see Remark 15. For a given component the product of the eigenvalues $\sigma_{k,j}$ with multiplicities divided by q equals one and the same root of unity of q -th order.

Proof of the lemma:

If there exists a $(p+1)$ -tuple of matrices A_j (or M_j) with a trivial centralizer for some $\lambda_0 \in L$, then there exist $(p+1)$ -tuples of matrices A_j (or M_j) with trivial centralizers for all $\lambda \in L$ sufficiently close to λ_0 (it suffices to apply the basic technical tool with diagonal matrices V_j which are polynomials of the semi-simple parts of the matrices G_j). The set of all such values λ is constructible. Hence, its intersection with L'_0 contains a Zariski open dense subset of L'_0 .

The lemma is proved. \square

3 Levelt's result and non-resonant eigenvalues

3.1 Levelt's result

In [L] Levelt gives the structure of the solution to a regular system at a pole:

Theorem 27 In a neighbourhood of a pole the solution to the regular linear system

$$\dot{X} = A(t)X \quad (9)$$

is representable in the form

$$X = U_j(t - a_j)(t - a_j)^{D_j}(t - a_j)^{E_j}G_j \quad (10)$$

where U_j is holomorphic in a neighbourhood of the pole a_j , with $\det U_j \neq 0$ for $t \neq a_j$; $D_j = \text{diag}(\varphi_{1,j}, \dots, \varphi_{n,j})$, $\varphi_{n,j} \in \mathbf{Z}$; $G_j \in GL(n, \mathbf{C})$. The matrix E_j is in upper-triangular form and the real parts of its eigenvalues belong to $[0, 1)$ (by definition, $(t - a_j)^{E_j} = e^{E_j \ln(t - a_j)}$). The numbers $\varphi_{k,j}$ satisfy the condition (12) formulated below.

System (9) is fuchsian at a_j if and only if

$$\det U_j(0) \neq 0 \quad (11)$$

We formulate the condition on $\varphi_{k,j}$. Let E_j have one and the same eigenvalue in the rows with indices $s_1 < s_2 < \dots < s_q$. Then we have

$$\varphi_{s_1,j} \geq \varphi_{s_2,j} \geq \dots \geq \varphi_{s_q,j} \quad (12)$$

Remarks 28 1) Denote by $\beta_{k,j}$ the diagonal entries (i.e. the eigenvalues) of the matrix E_j . If the system is fuchsian, then the sums $\beta_{k,j} + \varphi_{k,j}$ are the eigenvalues $\lambda_{k,j}$ of the matrix-residuum A_j , see [Bo1], Corollary 2.1.

2) The numbers $\varphi_{k,j}$ are defined as valuations in the solution eigensubspace for the eigenvalue $\exp(2\pi i \beta_{k,j})$ of the monodromy operator, see the details in [L].

3) One can assume without loss of generality that equal eigenvalues of E_j occupy consecutive positions on the diagonal and that the matrix E_j is block-diagonal, with diagonal blocks of sizes equal to their multiplicities. The diagonal blocks themselves are upper-triangular.

4) The following proposition will be used several times in the proofs and is of independent interest. It deals with the case when the monodromy group of a fuchsian system is reducible, i.e. there is a proper subspace invariant for all monodromy operators.

Proposition 29 *The sum of the eigenvalues $\lambda_{k,j}$ of the matrices-residua A_j of system (1) corresponding to an invariant subspace of the solution space is a non-positive integer.*

The proposition is proved in [Bo1], see Lemma 3.6 there.

3.2 Non-resonant eigenvalues

Definition 30 *Define as non-resonant the eigenvalues $\lambda_{k,j}$ of the conjugacy class $c_j \in gl(n, \mathbb{C})$ if there are no non-zero integer differences between them.*

Remark 31 *Let the PMV of the eigenvalues $\sigma_{k,j}$ of the monodromy operators M_j of system (1) be non-simple; denote by d^* the greatest common divisor of its components. Set $\lambda_{k,j} = \beta_{k,j} + \varphi_{k,j}$ for the eigenvalues of the matrices-residua A_j where $\text{Re}\beta_{k,j} \in [0, 1)$ and $\varphi_{k,j} \in \mathbb{Z}$, see Subsection 3.1. These conditions define unique numbers $\beta_{k,j}$. If d^* does not divide the sum $\sum_{j=1}^{p+1} \sum_{k=1}^n \beta_{k,j}$ (this sum is always integer because $\varphi_{k,j} \in \mathbb{Z}$ and there holds (5)), then the monodromy group cannot be realized by a fuchsian system with non-resonant eigenvalues because for such eigenvalues equality (5) would not hold.*

Lemma 32 *For non-resonant eigenvalues $\lambda_{k,j}$ of the matrix-residuum A_j of system (1) the Jordan normal forms of the matrix A_j and of the monodromy operator M_j are the same and M_j is conjugate to $\exp(2\pi i A_j)$.*

Proof:

Use Levelt's form (10) of the solution to system (9) (presumed to be fuchsian at a_j) and 1) and 3) from Remarks 28. One has $A(t) = \dot{X}X^{-1}$. Hence, if the eigenvalues of A_j are non-resonant, then to equal eigenvalues of E_j there correspond equal eigenvalues of D_j , the matrices D_j and E_j commute and $A_j = U_j(0)(D_j + E_j)(U_j(0))^{-1}$ (to be checked directly). One has $M_j = G_j^{-1} \exp(2\pi i E_j) G_j$. Hence, for the Jordan normal form $J(M_j)$ of M_j one has $J(M_j) = J(E_j) = J(D_j + E_j) = J(A_j)$.

The lemma is proved. \square

Lemma 33 *Every irreducible monodromy group with a simple PMV of the eigenvalues $\sigma_{k,j}$ can be realized by a fuchsian system with non-resonant eigenvalues $\lambda_{k,j}$.*

The lemma follows directly from Lemma 10 from [Ko4].

Corollary 34 *For a given $(p+1)$ -tuple of Jordan normal forms with a simple PMV the DSP is solvable for some generic eigenvalues for matrices A_j if and only if it is solvable for some generic eigenvalues for matrices M_j .*

Proof:

If for the given $(p + 1)$ -tuple of Jordan normal forms the DSP is solvable for some generic eigenvalues for matrices A_j , then one can choose such a $(p + 1)$ -tuple with not only generic, but with strongly generic non-resonant eigenvalues (this can be achieved by multiplying the given $(p + 1)$ -tuple of matrices A_j by some constant $c \in \mathbf{C}^*$), and then use Lemma 32 (if the PMV of the eigenvalues $\lambda_{k,j}$ is simple, then non-resonant eigenvalues $\lambda_{k,j}$ exist).

If for the given $(p + 1)$ -tuple of Jordan normal forms with a simple PMV the DSP is solvable for some generic eigenvalues for matrices M_j , then it is possible to realize such a monodromy group by a fuchsian system with non-resonant eigenvalues $\lambda_{k,j}$, see Lemma 33. Hence, for all j one would have $J(A_j) = J(M_j)$ (Lemma 32). \square

4 How to reduce the problem to the case of diagonalizable matrices ?

4.1 Correspondence between Jordan normal forms

All Jordan matrices in this subsection are presumed upper-triangular.

Definition 35 Let a non-diagonal Jordan normal form $J_0 = \{b_{i,l}\}$ of size n be given. We define its associated semi-simple Jordan normal form J_1 (also of size n) such that the quantities $r_j := r(J_j)$ and $d_j := d(J_j)$ are the same for $j = 0$ and $j = 1$.

A semi-simple Jordan normal form is the same as a partition of n , the parts being the multiplicities of the eigenvalues. If $J_0 = \{b_{i,l}\}$ ($i \in I_l$, $l \in L$), one views for each l the set $\{b_{i,l}\}$ as a partition of $\sum_i b_{i,l}$ and one takes for J_1 the disjoint sum of the dual partitions.

We will also say that the Jordan normal form J_1 corresponds to J_0 and that J_0 corresponds to J_1 . Any Jordan normal form J corresponding to J_1 corresponds to J_0 and J_0 corresponds to J .

Definition 36 Denote by G^0 , G^1 two Jordan matrices with Jordan normal forms J_0 , J_1 corresponding to each other, where G^1 is diagonal and G^0 is block-diagonal, each diagonal block having a single eigenvalue, different blocks having different eigenvalues. In the block-decomposition defined by the multiplicities of the eigenvalues of G^0 the matrix G^1 has diagonal blocks with mutually different eigenvalues. For each diagonal block the eigenvalues of G^1 occupying the last but q positions of the Jordan blocks of G^0 are equal (we denote them by h_q); for $q_1 \neq q_2$ one has $h_{q_1} \neq h_{q_2}$.

Example 37 Let $J_0 = \{\{4, 3, 2\}\{3, 1\}\}$, i.e. there are two eigenvalues to the first (resp. the second) of which there correspond Jordan blocks of sizes 4, 3, 2 (resp. 3, 1). Hence, J_1 is defined by the MV $(3, 3, 2, 2, 1, 1, 1)$. Indeed, the partition of $9 = 4 + 3 + 2$ dual to 4, 3, 2 is 3, 3, 2, 1, the partition of $4 = 3 + 1$ dual to 3, 1 is 2, 1, 1. When taking the direct sum of these dual partitions one rearranges the components of the MV so that they form a non-increasing sequence.

Remarks 38 1) If J_0 is the Jordan normal form of A , then the multiplicities of the eigenvalues for J_1 are the numbers $\dim \text{Ker}((A - \lambda I)^{k+1}) - \dim \text{Ker}((A - \lambda I)^k)$ which are non-zero.

2) One can show that any generic deformation of a matrix with Jordan normal form J_0 contains matrices with Jordan normal form J_1 and that these are the diagonalizable matrices from orbits of least dimension encountered in the deformation.

3) If a Jordan normal form is a direct sum of two Jordan normal forms with no eigenvalue in common, i.e. $J = J^* \oplus J^{**}$, and if the Jordan normal forms $J^{*'}$, $J^{**'}$ with no eigenvalue in common correspond to J^* , J^{**} , then $J^{*'} \oplus J^{**'}$ corresponds to J .

Proposition 39 *The quantities $r(J^{(i)})$ computed for two Jordan normal forms J' and J'' corresponding to one another coincide.*

Proof:

It suffices to prove the proposition for $J' = J_0$, $J'' = J_1$ (see Definition 35), i.e. to prove that $r_0 = r_1$. Let m^0 be the greatest number of Jordan blocks of J_0 with a given eigenvalue. Hence, $r_0 = n - m^0$. The construction of J_1 implies that the greatest of the multiplicities of the eigenvalues of J_1 equals m^0 – this follows from the definition of a dual partition. Thus, $r_1 = r_0$. \square

Proposition 40 *The dimensions of the orbits of two matrices with Jordan normal forms corresponding to one another are the same.*

Proof:

1⁰. It suffices to prove the proposition in the case when one of the Jordan normal forms is diagonal. Denote the two matrices by G^0 and G^1 where G^i are defined by Definition 36. The dimension of the orbit of G^j , $j = 0, 1$, equals $n^2 - \dim Z(G^j)$ where $Z(G^j)$ is the centralizer of G^j in $gl(n, \mathbf{C})$. Block-decompose the matrices from $gl(n, \mathbf{C})$ with sizes of the diagonal blocks equal to the multiplicities of the eigenvalues of G^0 . Then the off-diagonal blocks of $Z(G^j)$ are 0; indeed, two diagonal blocks of G^j ($j = 1, 2$) have no eigenvalue in common. This observation allows when computing the dimensions of the orbits to consider only the case when J_0 has only one eigenvalue.

2⁰. Show that in this case one has $\dim Z(G^0) = \dim Z(G^1)$, hence, the dimensions of the orbits of G^0 and G^1 are the same. One has

$$\dim Z(G^0) = b_1 + 3b_2 + 5b_3 + \dots + (2r - 1)b_r$$

where $b_1 \geq b_2 \geq b_3 \geq \dots \geq b_r$ are the sizes of the Jordan blocks of J_0 , see [Ar], p. 229;

$$\dim Z(G^1) = (k_1)^2 + (k_2)^2 + \dots + (k_{n_1})^2$$

where k_i are the multiplicities of the eigenvalues of J_1 .

3⁰. The first b_r of the numbers k_j equal r , the next $(b_{r-1} - b_r)$ equal $r - 1$, the next $(b_{r-2} - b_{r-1})$ equal $r - 2$ etc. Thus

$$\begin{aligned} (k_1)^2 + \dots + (k_{n_1})^2 &= b_r r^2 + (b_{r-1} - b_r)(r - 1)^2 + \dots + (b_1 - b_2) \times 1^2 = \\ &= b_r[r^2 - (r - 1)^2] + b_{r-1}[(r - 1)^2 - (r - 2)^2] + \dots + b_1 \times 1 = \\ &= (2r - 1)b_r + (2r - 3)b_{r-1} + \dots + b_1 = \dim Z(G^0) \end{aligned}$$

The proposition is proved. \square

Proposition 41 *Let the two Jordan normal forms $J^{n'}$ and $J^{n''}$ correspond to one another. Choose in each of them an eigenvalue with maximal number of Jordan blocks. By Proposition 39 these numbers coincide. Denote them by k' . Decrease by 1 the sizes of the k smallest Jordan blocks with these eigenvalues where $k \leq k'$. Then the two Jordan normal forms of size $n - k$ obtained in this way correspond to one another.*

Remark 42 *The proposition implies in particular that if a Jordan normal form has several (say, s) eigenvalues with the maximal number k' of Jordan blocks and if one constructs s new Jordan normal forms by decreasing by 1 the sizes of the k smallest blocks with a given one of these eigenvalues, then these s Jordan normal forms correspond to one another.*

Proof:

1⁰. It suffices to prove the proposition in the case when the Jordan normal form $J^{n''}$ is diagonal (if this is not so, then consider together with $J^{n'}$ and $J^{n''}$ the diagonal Jordan normal form $J^{n'''} corresponding to them, then prove the proposition for the couples $J^{n'}, J^{n'''}$ and $J^{n''}, J^{n'''}$). In the case when $J^{n''}$ is diagonal one simply decreases by k the biggest component of the MV.$

2⁰. Assume that the ν -th eigenvalue of $J^{n'}$ has k' Jordan blocks, of sizes $b_{i,\nu}$, $i = 1, \dots, k'$. Decreasing by 1 the least k of the integers $b_{i,\nu}$ (considered as parts of the partition of $\sum_{i=1}^{k'} b_{i,\nu}$) results in decreasing by k the biggest part of its dual partition. By definition, this biggest part equals k' and it is (one of) the biggest component(s) of the MV defining $J^{n''}$.

The proposition is proved. \square

Proposition 43 *For each diagonal Jordan normal form J_1 there exists a unique Jordan normal form J_0 with a single eigenvalue which corresponds to J_1 . Hence, the same is true for any Jordan normal form.*

Proof:

It follows from the construction of J_1 after J_0 that if the multiplicities of the eigenvalues of J_1 equal $g_1 \geq \dots \geq g_d$, $g_1 + \dots + g_d = n$, then J_0 has exactly g_ν Jordan blocks of size $\geq \nu$. This condition defines a unique Jordan normal form J_0 with a single eigenvalue. \square

Recall that the matrices G^0 and G^1 were defined in Definition 35.

Proposition 44 1) *If G^0 is nilpotent, then the orbits of the matrices εG^1 and $G^0 + \varepsilon G^1$ are the same for $\varepsilon \in \mathbf{C}^*$.*

2) *If G^0 is not necessarily nilpotent, then the matrix $G^0 + \varepsilon G^1$ is diagonalizable and for $\varepsilon \in \mathbf{C}^*$ small enough its Jordan normal form is J_1 , its orbit is the one of $G_s^0 + \varepsilon G^1$ where G_s^0 is the semisimple part of G^0 .*

Proof:

1⁰. Let G^0 be nilpotent (hence, there is just one diagonal block of size n). Conjugate the matrices εG^1 and $G^0 + \varepsilon G^1$ with a permutation matrix Q such that after the permutation the eigenvalues h_0 occupy the last positions on the diagonal preceded by the eigenvalues h_1 preceded by the eigenvalues h_2 etc.

2⁰. If one block-decomposes a matrix with sizes of the diagonal blocks equal to the multiplicities of the eigenvalues h_q , then the units of the matrix $Q^{-1}G^0Q$ will be all in the blocks above the diagonal. Hence, the matrix $G^* = Q^{-1}(G^0 + \varepsilon G^1)Q$ in this block decomposition is block upper-triangular and has scalar diagonal blocks with mutually distinct eigenvalues. Hence, one can conjugate this matrix with a block upper-triangular matrix and after the conjugation the

units above the diagonal disappear and the resulting matrix is diagonal, with the same diagonal blocks as G^* . Hence, this is the matrix εG^1 .

3⁰. If G^0 has one eigenvalue (not necessarily equal to 0), then the second statement of the proposition follows from the first one.

4⁰. If G^0 is arbitrary, then one can block decompose it, the diagonal blocks having each one eigenvalue, the eigenvalues of different diagonal blocks being different, and then apply the result from 3⁰ to every diagonal block. (For small values of $\varepsilon \in \mathbf{C}^*$ two different diagonal blocks will have no eigenvalue in common.)

The proposition is proved. \square

Proposition 45 *Denote by $J = \{b_{i,l}\}$ an arbitrary Jordan normal form of size n and by J' its corresponding Jordan normal form with a single eigenvalue. Recall that for each fixed l one has $b_{1,l} \geq b_{2,l} \geq \dots \geq b_{s_l,l}$. Then the size of the k -th Jordan block of J' (in decreasing order) equals $\sum_l b_{k,l}$ (if some of the numbers participating in this sum are not defined, then they are presumed to equal 0).*

Proof:

1⁰. It suffices to consider the case of two eigenvalues. The general case can be treated by induction on the number of eigenvalues (one represents a Jordan normal form J_1^n with k eigenvalues as direct sum of a Jordan normal form J_2^m with a single eigenvalue and a Jordan normal form J_3^{n-m} with $k-1$ eigenvalues; then one finds the Jordan normal form J_4^{n-m} with a single eigenvalue corresponding to J_3^{n-m} and finally the Jordan normal form J_5^n with a single eigenvalue corresponding to $J_2^m \oplus J_4^{n-m}$; J_5^n corresponds to J_1^n and $J_2^m \oplus J_4^{n-m}$ has two eigenvalues).

In the case of two eigenvalues it suffices to show that if the sizes of the blocks of J' are as in the proposition, then to J and J' there corresponds one and the same diagonal Jordan normal form $J_d = J'_d$.

2⁰. Denote the two eigenvalues of J by λ and σ . Assume that to λ there correspond no less Jordan blocks than to σ , i.e. $s_1 \geq s_2$. This means that the greatest of the multiplicities of eigenvalues both of J_d and of J'_d equals s_1 .

Consider the last $b_{s_1,1}$ positions of every Jordan block with eigenvalue λ . They give rise to $b_{s_1,1}$ eigenvalues each of multiplicity s_1 in J_d .

Decrease

- 1) the size n of the matrices by $s_1 b_{s_1,1}$,
- 2) the sizes of each of the Jordan blocks of J with eigenvalue λ by $b_{s_1,1}$ and
- 3) the sizes of all Jordan blocks of J' by $b_{s_1,1}$.

Hence, in each of the diagonal Jordan normal forms J_d and J'_d one loses $b_{s_1,1}$ eigenvalues each of multiplicity s_1 . Hence, the Jordan normal forms J_d and J'_d coincide or not simultaneously before and after the reduction by $s_1 b_{s_1,1}$ of the sizes of the matrices.

3⁰. After a finite number of such reductions one of the two eigenvalues becomes of multiplicity 0; in this case there is nothing to prove.

The proposition is proved. \square

Denote by J and J' an arbitrary Jordan normal form and its corresponding Jordan normal form with a single eigenvalue. Consider a couple Δ, D' of Jordan matrices with these Jordan normal forms D' being nilpotent. Suppose that the Jordan blocks of Δ of sizes $b_{k,l}$ for k fixed are situated in the same rows where is situated the Jordan block of size $\sum_l b_{k,l}$ of D' (see the previous proposition). Denote by Δ_s the diagonal (i.e. semi-simple) part of the matrix Δ .

Proposition 46 *For all $\varepsilon \neq 0$ the matrix $\varepsilon \Delta_s + D'$ is conjugate to $\varepsilon \Delta$.*

It suffices to prove the proposition in the case when D' has a single Jordan block of size n . In this case one checks directly that for all eigenvalues a of Δ one has $\text{rk}(\varepsilon(\Delta_s - aI) + D') = \text{rk}(\varepsilon(\Delta - aI)) = n - 1$. For all other values of $a \in \mathbf{C}$ these ranks equal n . \square

Remark 47 *Permute the diagonal entries of Δ_s so that before and after the permutation each entry remains in one of the rows of one and the same Jordan block of D' . Then the proposition holds again and the proof is the same.*

4.2 Subordinate conjugacy classes and normalized chains of eigenvalues

Definition 48 *Given two conjugacy classes c' , c'' with one and the same eigenvalues, of one and the same multiplicities, we say that c'' is subordinate to c' if c'' lies in the closure of c' , i.e. for any matrix $A \in c''$ there exists a deformation $\tilde{A}(\varepsilon)$, $\tilde{A}(0) = A$ such that for $0 \neq \varepsilon \in (\mathbf{C}, 0)$ one has $\tilde{A}(\varepsilon) \in c'$.*

Example 49 *Let $n = 4$ and let the eigenvalues be a, a, b, b , $a \neq b$. Let c' (resp. c'') have one Jordan block 2×2 (resp. two Jordan blocks 1×1) with eigenvalue a and both c' and c'' have two Jordan blocks 1×1 with eigenvalue b . Then c'' is subordinate to c' . If the conjugacy class c''' has the same eigenvalues, two Jordan blocks 1×1 with eigenvalue a and one Jordan block 2×2 with eigenvalue b , then neither c' is subordinate to c''' nor c''' is subordinate to c' (and c'' is subordinate to c''').*

Notice that in the above example the Jordan normal forms of c' and c''' are the same.

Definition 50 *Given two Jordan normal forms J' , J'' , we say that J'' is subordinate to J' if there exist conjugacy classes c' , c'' defining the Jordan normal forms J' , J'' such that c'' is subordinate to c' .*

Definition 51 1) *For a diagonalizable matrix represent the set of its eigenvalues as a union of maximal non-intersecting subsets of eigenvalues congruent modulo \mathbf{Z} (called further \mathbf{Z} -subsets). For each \mathbf{Z} -subset define its multiplicity vector where the different eigenvalues of the \mathbf{Z} -subset are ordered so that their real parts form a decreasing sequence. Then the eigenvalues of the matrix are said to form a normalized chain if for every such multiplicity vector its components form a non-decreasing sequence.*

2) *If the eigenvalues of the diagonalizable matrix A form a normalized chain, then the multiplicity vector of each \mathbf{Z} -subset defines a diagonal Jordan normal form J_i . Denote by J'_i its corresponding Jordan normal form with a single eigenvalue. Denote by $\tilde{J}(A)$ the Jordan normal form $\oplus_i J'_i$ where the sum is taken over all \mathbf{Z} -subsets; for $i_1 \neq i_2$ the eigenvalues of J'_{i_1} and J'_{i_2} are different. Hence, $\tilde{J}(A)$ corresponds to $J(A)$, see 3) of Remarks 38. If $J(A) = J^0$, then we set $\tilde{J}(J^0) = \tilde{J}(A)$.*

Lemma 52 *Let in system (1) the matrix A_1 be with Jordan normal form J_1 and let its eigenvalues form a normalized chain. Then the Jordan normal form of the monodromy operator M_1 is either $J^* := \tilde{J}(J_1)$ or is one subordinate to it.*

Proof:

1⁰. Consider first the case when A_1 is diagonal and J^* has just one eigenvalue. Use Theorem 27. If the solution to system (1) is represented in form (10), with $\varphi_{1,1} \geq \dots \geq \varphi_{n,1}$, then one has $E_1 = \alpha I + F$ ($\text{Re}(\alpha) \in [0, 1)$) where the matrix F is nilpotent and upper-triangular.

2⁰. More exactly, F is block upper-triangular, with zero diagonal blocks; the diagonal blocks are of sizes equal to the multiplicities of the eigenvalues of the matrix D_1 from (10). Indeed, the presence of non-zero entries in the diagonal blocks of F would result in A_1 not being diagonalizable (we propose to the reader to check this oneself).

3⁰. Denote the MV of the eigenvalues of D_1 (it is also the one of J_1) by (l_d, \dots, l_1) where $l_d \leq \dots \leq l_1$ (these inequalities follow from the definition of J_1 in the previous subsection – for each q the number l_q of eigenvalues h_q equals the number of Jordan blocks of J^* of size $\geq q+1$).

4⁰. The rank of the matrix $(F)^\nu$ cannot exceed $\tilde{l}_\nu := l_d + \dots + l_{\nu+1}$ (only the first \tilde{l}_ν rows of $(F)^\nu$ can be non-zero). This is exactly the rank of $(N)^\nu$, N being a nilpotent matrix with Jordan normal form J^* . Hence, the Jordan normal form of F is either J_0 or is one subordinate to it. Indeed, the inequalities $\text{rk}(F)^\nu \leq \text{rk}(N)^\nu$, $\nu = 1, 2, \dots$ imply that either the orbits of F and N coincide (if there are equalities everywhere) or that the orbit of F lies in the closure of the one of N (if at least one inequality is strict), see [Kr], p. 21.

5⁰. On the other hand, one has (up to conjugacy) $M_1 = \exp(2\pi i E_1) = \exp(2\pi i \alpha) \exp(2\pi i F)$. This means that the Jordan normal forms of M_1 and F coincide. Hence, the Jordan normal form of M_1 is either J^* or is one subordinate to it.

6⁰. In the general case (when J^* has several eigenvalues) one uses 3) of Remarks 28 and applies the above reasoning to each diagonal block of E_j , i.e. to each eigenvalue of the monodromy operator M_j .

The lemma is proved. \square

4.3 Reduction to the case of diagonalizable matrices A_j

Denote by J_j^0 the Jordan normal forms of the matrices A_j or M_j . Denote by J_j^1 their corresponding diagonal Jordan normal forms defined in Subsection 4.1.

Lemma 53 1) *The DSP is solvable for Jordan normal forms J_j^0 with a simple PMV and for some generic eigenvalues if and only if it is solvable for the Jordan normal forms J_j^1 and for some generic eigenvalues.*

2) *If for some generic eigenvalues and given Jordan normal forms J_j with a simple PMV the DSP is solvable, then it is solvable (for some generic eigenvalues) for all $(p+1)$ -tuples of Jordan normal forms J'_j where for each j either J_j is subordinate to J'_j or $J_j = J'_j$.*

The lemma holds for matrices A_j and for matrices M_j .

Proof:

1⁰. Prove the lemma first for matrices A_j . Denote by G_j^i , $i = 0, 1$, two Jordan matrices defining the same Jordan normal forms as J_j^i and such that $A_j = Q_j^{-1} G_j^0 Q_j$; we define the matrices G_j^i like the matrices G^i from Definition 36. The existence of irreducible $(p+1)$ -tuples of matrices

$$\tilde{A}_j = (I + \varepsilon X_j(\varepsilon))^{-1} Q_j^{-1} (G_j^0 + \varepsilon G_j^1) Q_j (I + \varepsilon X_j(\varepsilon)),$$

satisfying (2), with $\varepsilon \in (\mathbf{C}, 0)$ is proved using the basic technical tool, see Subsection 2.1. Hence, for $\varepsilon \neq 0$ small enough the Jordan normal form of \tilde{A}_j is J_j^1 (and \tilde{A}_j is conjugate to $(G_j^0 + \varepsilon G_j^1)$, see Proposition 44). For these values of ε the eigenvalues of \tilde{A}_j will still be generic.

Thus the existence of $(p+1)$ -tuples with Jordan normal forms J_j^0 implies the existence of ones with Jordan normal forms J_j^1 .

2⁰. By analogy one proves that the existence of irreducible $(p+1)$ -tuples of matrices A_j for the $(p+1)$ -tuple of Jordan normal forms J_j (and for some generic eigenvalues) implies the one for the $(p+1)$ -tuple of Jordan normal forms J_j' (and for some generic eigenvalues) where for each j either $J_j' = J_j$ or J_j is subordinate to J_j' . To this end one looks for the new $(p+1)$ -tuple of matrices

$$\tilde{A}_j = (I + \varepsilon X_j(\varepsilon))^{-1} Q_j^{-1} (G_j + \varepsilon V_j(\varepsilon)) Q_j (I + \varepsilon X_j(\varepsilon))$$

(where G_j are Jordan matrices with Jordan normal forms J_j and the matrices V_j (holomorphic in $\varepsilon \in (\mathbf{C}, 0)$) are chosen such that \tilde{A}_j have for $\varepsilon \neq 0$ Jordan normal form J_j'). The possibility to choose such matrices V_j follows from the definitions of subordinate orbits and subordinate Jordan normal forms. This proves part 2) of the lemma for matrices A_j .

3⁰. Assume that the Jordan matrices G_j^i have the same meaning as in 1⁰. Choose strongly generic eigenvalues of G_j^1 such that for every j they form a normalized chain (see the previous subsection). Suppose that there exists a fuchsian system (1) with A_j conjugate to G_j^1 (for all j). Then for every j the Jordan normal form $J_j'^0$ of the monodromy operator M_j is either J_j^0 or is a Jordan normal form subordinate to J_j^0 (Lemma 52). Such an irreducible monodromy group can be realized by a fuchsian system whose matrices-residua have Jordan normal forms $J_j'^0$ (by Lemma 33 – such matrices-residua correspond to a non-resonant choice of the eigenvalues $\lambda_{k,j}$). By 2) of the present lemma, there exist such $(p+1)$ -tuples of matrices-residua also for Jordan normal forms J_j^0 . This proves part 1) of the lemma for matrices A_j .

4⁰. Having proved the lemma for matrices A_j , one knows from Corollary 34 that it is true for matrices M_j as well.

The lemma is proved. \square

5 The basic theorem for diagonalizable matrices

Definition 54 *A simple PMV is called good if the DSP is solvable for some generic eigenvalues and for the $(p+1)$ -tuple of diagonal Jordan normal forms defined by the PMV. For $n = 1$ the only possible PMV is also defined as good.*

For the PMV $\Lambda^n = (\Lambda_1^n, \dots, \Lambda_{p+1}^n)$ (where $\Lambda_j^n = (m_{1,j}, \dots, m_{k_j,j})$, $m_{1,j} + \dots + m_{k_j,j} = n$) we presume that the following condition holds:

$$m_{1,j} \geq \dots \geq m_{k_j,j} \quad (*_n)$$

Hence, $r_j = n - m_{1,j}$.

Lemma 55 *A simple PMV satisfying condition (ω_n) is good.*

Proof:

1⁰. We use Definition 12 and Theorem 13. For any $(p+1)$ -tuple of diagonal Jordan normal forms J_j^1 one can find the $(p+1)$ -tuple of corresponding Jordan normal forms with a single eigenvalue J_j^0 , see Proposition 43. If the PMV of the Jordan normal forms J_j^1 is simple, then the Jordan normal forms J_j^0 do not correspond to any of the four exceptional cases cited in Theorem 13. If condition (ω_n) holds for the Jordan normal forms J_j^1 , then it holds for the Jordan normal forms J_j^0 as well (Proposition 39).

²⁰. Hence, there exist nice $(p+1)$ -tuples of nilpotent matrices A_j or of unipotent matrices M_j with Jordan normal forms J_j^0 , see Definition 12 and Theorem 13. The existence of irreducible $(p+1)$ -tuples of matrices with Jordan normal forms J_j^1 is deduced from the existence for J_j^0 by means of the basic technical tool, see Subsections 2.1 and 2.2, by complete analogy with ¹⁰ of the proof of Lemma 53.

The lemma is proved. \square

For a given simple PMV Λ^n define the numbers $n = n_0 > n_1 > \dots > n_s$ like this was done before Theorem 17 by means of the map Ψ (in our particular case of diagonalizable matrices we operate over PMVs instead of Jordan normal forms): if Λ^n satisfies condition (ω_n) or if it does not satisfy condition (β_n) or if $n = 1$, then set $s = 0$. If not, then set $n_1 = r_1 + \dots + r_{p+1} - n$. Hence, $n_1 < n$ (otherwise Λ^n satisfies condition (ω_n)).

Define the PMV $\Lambda^{n_1} = (\Lambda_1^{n_1}, \dots, \Lambda_{p+1}^{n_1})$. Set $\Lambda_j^{n_1,0} = (m_{1,j} - n + n_1, m_{2,j}, \dots, m_{k_j,j})$ (recall that $m_{1,j} - n + n_1 = n_1 - r_1 \geq 0$ because there hold conditions $(*_n)$ and (β_n)). For each j rearrange the components of $\Lambda_j^{n_1,0}$ to obtain condition $(*_{n_1})$ – this gives the MVs $\Lambda_j^{n_1}$.

Suppose that the PMVs Λ^{n_i} are constructed for $i = 0, \dots, k$. If Λ^{n_k} satisfies condition (ω_{n_k}) or if it does not satisfy condition (β_{n_k}) or if $n_k = 1$, then set $s = k$. If not, then define n_{k+1} and $\Lambda^{n_{k+1}}$ after n_k and Λ^{n_k} in the same way as n_1 and Λ^{n_1} were defined after n and Λ^n etc. In the end we have either $n_s = 1$ or Λ^{n_s} satisfies condition (ω_{n_s}) or it does not satisfy condition (β_{n_s}) .

Remarks 56 1) Given a $(p+1)$ -tuple of arbitrary Jordan normal forms J_j^n , construct the PMV Λ^n of the $(p+1)$ -tuple of diagonal Jordan normal forms corresponding to them. Hence, the quantities r_j and n_1 defined for both $(p+1)$ -tuples coincide (this can be deduced from Proposition 39). By Proposition 41, the PMV Λ^{n_1} defines the $(p+1)$ -tuple of diagonal Jordan normal forms corresponding to the Jordan normal forms $J_j^{n_1}$. In the same way one shows that the numbers s and n_1, \dots, n_s are the same when defined for the $(p+1)$ -tuple of Jordan normal forms J_j^n and when defined for the PMV Λ^n and that for all ν the PMVs Λ^{n_ν} define the $(p+1)$ -tuples of diagonal Jordan normal forms corresponding to $J_j^{n_\nu}$.

2) For diagonal Jordan normal forms the PMVs Λ^{n_ν} do not depend on the choice of eigenvalue in the map Ψ . This together with Remark 42 explains why Theorem 17 is true whichever choice of eigenvalue is made in Ψ – for two such choices the two respective Jordan normal forms $J_j^{n_\nu}$ will correspond to one another for all j and ν (and the sizes n_ν will be the same) because they correspond to one and the same PMVs Λ^{n_ν} . Hence, the PMV Λ^{n_s} and the $(p+1)$ -tuple of Jordan normal forms $J_j^{n_s}$ satisfy or not condition (ω_n) (resp. (β_n)) simultaneously.

Lemma 57 If the PMV Λ^{n_ν} is simple, then the PMV $\Lambda^{n_{\nu+1}}$ is also simple.

Proof:

We prove the lemma for $\nu = 0$, for arbitrary ν it is proved by analogy. Suppose that Λ^{n_1} is non-simple. Then for every j the greatest common divisor l of its components divides $m_{2,j}, \dots, m_{k_j,j}$ and $m_{1,j} - n + n_1$, hence, it divides n_1 (the length of Λ^{n_1}). But $n_1 = r_1 + \dots + r_{p+1} - n$ and l divides r_j (because $r_j = m_{2,j} + \dots + m_{k_j,j}$); hence, l divides n and $m_{1,j}$ (because $m_{1,j} = n - r_j$). This means that Λ^n is non-simple – a contradiction. \square

Theorem 58 A simple PMV Λ^n is good if and only if it satisfies condition (β_n) and either the PMV Λ^{n_s} defined above satisfies condition (ω_{n_s}) or one has $n_s = 1$. The theorem is true both in the additive and in the multiplicative version of the DSP.

Definition 59 For a given $(p+1)$ -tuple of Jordan normal forms J_j^n with $d_j = d(J_j^n)$ call the quantity $\kappa := 2n^2 - d_1 - \dots - d_{p+1}$ index of rigidity of the $(p+1)$ -tuple. (This notion was introduced by N.Katz in [Ka].) For a PMV define its index of rigidity as the one of the $(p+1)$ -tuple of diagonal Jordan normal forms defined by it.

Lemma 60 The PMVs Λ^{n_ν} and $\Lambda^{n_{\nu+1}}$ have the same index of rigidity. In particular, they satisfy or not the respective conditions (α_{n_ν}) and $(\alpha_{n_{\nu+1}})$ simultaneously.

Proof: We prove the lemma for $\nu = 0$, for arbitrary ν the proof is analogous. Set $d_j = d(\Lambda_j^n)$, $d_j^1 = d(\Lambda_j^{n_1})$. One has $d_j^1 = d_j - 2(n - n_1)r_j$ (by direct computation) and

$$\sum_{j=1}^{p+1} d_j^1 = \sum_{j=1}^{p+1} d_j - 2(n - n_1) \sum_{j=1}^{p+1} r_j = \sum_{j=1}^{p+1} d_j - 2(n - n_1)(n + n_1) = \sum_{j=1}^{p+1} d_j - 2n^2 + 2(n_1)^2$$

which shows that the index of rigidity remains the same. \square

Corollary 61 If the $(p+1)$ -tuple of Jordan normal forms J_j^n (not necessarily diagonal) satisfies the equality $\sum_{j=1}^{p+1} d_j = 2n^2 - 2 + \chi$, $\chi \geq 0$, then for the quantities $d_j^\nu = d(J_j^{n_\nu})$ (where the Jordan normal forms $J_j^{n_\nu}$ are defined before Theorem 17) one has $d^\nu := \sum_{j=1}^{p+1} d_j^\nu = 2(n_\nu)^2 - 2 + \chi$.

Indeed, one can define the PMV Λ^n of the eigenvalues of the diagonal Jordan normal forms corresponding to J_j^n and then the PMVs Λ^{n_ν} . For all ν the PMVs Λ^{n_ν} define diagonal Jordan normal forms corresponding to $J_j^{n_\nu}$, see Remarks 56. For the $(p+1)$ -tuples $\Lambda_j^{n_\nu}$ and $J_j^{n_\nu}$ the quantity d^ν is the same (Proposition 40). \square

Remark 62 The conditions $n_s > 1$ and (α_n) being a strict inequality are equivalent. Indeed, if $n_s > 1$, then condition (ω_{n_s}) holds for the $(p+1)$ -tuple of Jordan normal forms $J_j^{n_s}$. By Lemma 3 from [Ko1], inequality (α_{n_s}) holds for the Jordan normal forms $J_j^{n_s}$ and is strict. Corollary 61 allows to conclude that condition (α_n) is a strict inequality (i.e. $\chi > 0$).

Corollary 63 If the PMV Λ^n is simple and good, then so are the PMVs Λ^{n_ν} , $\nu = 1, \dots, s$.

The corollary follows from the definition of the PMVs Λ^{n_ν} , from Lemmas 60 and 57 and from Theorem 58.

Proof of Theorem 21:

Proposition 39 and Remarks 56 show that the Jordan normal forms J_j^n and $J_j^{n_s}$ satisfy conditions *i*) and *ii*) of Theorem 17 if and only if the PMVs Λ^n and Λ^{n_s} satisfy the conditions of Theorem 58 (where Λ^n defines the diagonal Jordan normal forms corresponding to the $(p+1)$ -tuple of Jordan normal forms J_j^n).

By Lemma 53, there exist for some generic eigenvalues matrices A_j or M_j satisfying (2) or (3) with Jordan normal forms J_j^n if and only if this is the case of diagonalizable matrices defined by the PMV Λ^n . Thus Theorem 21 results from Theorem 58. \square

6 Proof of the sufficiency in Theorem 58

6.1 Proof of the theorem itself

The lemmas from this subsection are proved in the next ones. We prove the sufficiency in the case of matrices A_j , for matrices M_j it follows then from Corollary 34.

Induction on n . For $n = 1$ and 2 the theorem is checked straightforwardly. If Λ^n satisfies condition (ω_n) , then Λ^n is good, see Lemma 55. If not, then Λ^{n_1} satisfies the conditions of the theorem with n replaced by n_1 (this follows from the definition of the PMVs Λ^{n_ν} before Remarks 56). By inductive assumption, there exist (for generic eigenvalues) irreducible $(p+1)$ -tuples of diagonalizable $n_1 \times n_1$ -matrices B_j (satisfying (2)) with PMV equal to Λ^{n_1} . We assume that the eigenvalue λ_1 of B_1 of multiplicity $m_{1,1} - n + n_1$ (when this multiplicity is not 0) equals 1 and that for $j > 1$ the eigenvalue λ_j of B_j of multiplicity $m_{1,j} - n + n_1$ equals 0. This can be achieved by replacing the matrices B_j by $B_j - \lambda_j I$, $j > 1$, and B_1 by $B_1 + (\lambda_2 + \dots + \lambda_{p+1})I$, and by multiplying all matrices by $c \in \mathbf{C}^*$.

For the sake of convenience we make a circular permutation of the components of the MVs Λ_j^n and $\Lambda_j^{n_1}$ putting their first components (i.e. $m_{1,j}$ and $m_{1,j} - n + n_1$) in last position.

Define the PMV $\tilde{\Lambda}^n$ as follows: for $j > 1$ set $\tilde{\Lambda}_j^n = \Lambda_j^n$; set $\tilde{\Lambda}_1^n = (m_{2,1}, \dots, m_{k_1,1}, m_{1,1} - n + n_1, n - n_1)$. Define the diagonalizable $n \times n$ -matrices A_j^0 with PMV $\tilde{\Lambda}^n$ as $A_j^0 = \begin{pmatrix} B_j & 0 \\ 0 & 0 \end{pmatrix}$, with B_j as above. (The multiplicity of 0 as eigenvalue of A_j^0 equals $m_{1,j}$ for $j > 1$ and $n - n_1$ for $j = 1$.) Construct a $(p+1)$ -tuple of matrices $A_j^1 = \begin{pmatrix} B_j & B_j Y_j \\ 0 & 0 \end{pmatrix}$, Y_j being $n_1 \times (n - n_1)$, such that the monodromy operator M_1^1 at a_1 of the fuchsian system

$$\dot{X} = \left(\sum_{j=1}^{p+1} A_j^1 / (t - a_j) \right) X \quad (13)$$

is diagonalizable (see Lemma 64 below); we set $Y_1 = 0$. Notice that for each j the matrix A_j^1 is conjugate to the matrix A_j^0 . We assume that the only couple of eigenvalues of some matrix A_j^1 whose difference is a non-zero integer are the eigenvalues 0 and 1 of A_1^1 . This is not restrictive, see Lemma 25.

We also assume $A_1^0 = A_1^1$ to be diagonal (hence, B_1 as well) and the eigenvalues of the matrices B_j to be generic.

Lemma 64 *The operator M_1^1 is diagonalizable if and only if the following conditions hold:*

$$\sum_{j=2}^{p+1} (A_j^1 / (a_j - a_1))_{\kappa, \nu} = 0 \quad , \quad \kappa = r_1 + 1, \dots, n_1 ; \nu = n_1 + 1, \dots, n \quad (14)$$

(double subscripts indicate matrix entries).

Remarks 65 1) The lemma is vacuous if $r_1 = n_1$ when there is no condition to verify and M_1^1 is automatically diagonalizable.

2) If at least one of the matrices Y_j , $j > 1$, is non-zero, then the $(p+1)$ -tuple of matrices A_j^1 is not conjugate to the $(p+1)$ -tuple of matrices A_j^0 . Indeed, if this were the case, then the conjugation should be carried out by a matrix commuting with $A_1^0 = A_1^1$, i.e. block-diagonal, with diagonal blocks of sizes $n_1 \times n_1$ and $(n - n_1) \times (n - n_1)$. Such a conjugation cannot annihilate the blocks $B_j Y_j$.

Definition 66 We say that the columns of the $(p+1)$ -tuple of $q \times r$ -matrices C_j are linearly independent if for no r -tuple of constants $\beta_i \in \mathbf{C}$ (not all of them being 0) one has $\sum_{i=1}^r \beta_i C_{j,i} = 0$ for $j = 1, \dots, p+1$ where $C_{j,i}$ is the i -th column of the matrix C_j . In the same way one defines independence of rows.

Definition 67 Denote by $\tilde{\mathbf{C}}^v$ the linear space of p -tuples of vectors $T_j \in \mathbf{C}^{n_1}$, $j = 2, \dots, p+1$, where

$$T_2 + \dots + T_{p+1} = 0 \quad (15)$$

and $T_j = B_j U_j$ for some $U_j \in \mathbf{C}^{n_1}$. Denote by $\tilde{\mathbf{C}}^w \subset \tilde{\mathbf{C}}^v$ its subspace satisfying the condition

$$(\alpha_2 T_2 + \dots + \alpha_{p+1} T_{p+1})|_\kappa = 0, \quad \alpha_j = 1/(a_j - a_1), \quad \kappa = r_1 + 1, \dots, n_1 \quad (16)$$

The notation $|_\kappa$ means the κ -th coordinate of the vector, see Lemma 64.

It is clear that $v \stackrel{\text{def}}{=} \dim \tilde{\mathbf{C}}^v \geq r_2 + \dots + r_{p+1} - n_1 = n - r_1$ (the image of the linear operator $\xi_j : \mathbf{C}^{n_1} \rightarrow \mathbf{C}^{n_1}$, $\xi_j : (\cdot) \mapsto B_j(\cdot)$ is of dimension r_j and equation (15) is equivalent to $\leq n_1$ linearly independent equations). In the same way one deduces the inequality $w \stackrel{\text{def}}{=} \dim \tilde{\mathbf{C}}^w \geq v - (n_1 - r_1) \geq n - n_1$.

Lemma 68 One has $v = n - r_1$ and $w = n - n_1$.

Lemma 69 There exists a $(p+1)$ -tuple of matrices Y_j such that

- 1) $Y_1 = 0$ and for $j > 1$ Y_j belongs to the image of the linear operator $\tau_j : (\cdot) \mapsto B_j(\cdot)$ acting on the space of $n_1 \times (n - n_1)$ -matrices;
- 2) $\sum_{j=2}^{p+1} B_j Y_j = 0$;
- 3) the monodromy operator M_1^1 at a_1 of the fuchsian system (13) is diagonalizable;
- 4) the columns of the p -tuple of matrices $B_j Y_j$, $j = 2, \dots, p+1$, are linearly independent; they are a basis of the space $\tilde{\mathbf{C}}^w$ defined above.

Lemma 70 1) The centralizer of the $(p+1)$ -tuple of matrices A_j^1 satisfying 1) – 4) of the previous lemma is trivial.

2) The centralizer of the monodromy group of system (13) (in which the $(p+1)$ -tuple of matrices A_j^1 satisfies 1) – 4) of Lemma 69) is trivial.

Lemma 70 is necessary for the proof of the following lemma from which follows the proof of the sufficiency.

Lemma 71 Denote by \mathcal{L} the set of eigenvalues of the $(p+1)$ -tuple of matrices M_j^1 . There exist (for generic eigenvalues close to \mathcal{L}) irreducible $(p+1)$ -tuples of diagonalizable matrices M_j satisfying (3), with PMV equal to Λ^n .

By Lemma 25 there exist such $(p+1)$ -tuples for all eigenvalues from a Zariski open dense subset of the set of all generic eigenvalues with multiplicities defined by Λ^n . Thus we have proved that in the multiplicative version the simple PMV Λ^n is good. It is good in the additive one as well due to Corollary 34.

The sufficiency is proved.

6.2 Proof of Lemma 64

The fuchsian system (13) represented by its Laurent series at a_1 looks like this:

$$\dot{X} = [A_1^1/(t - a_1) + B + o(1)]X, \quad B = -(\sum_{j=2}^{p+1} A_j^1/(a_j - a_1))$$

One can assume that $A_1^1 = \text{diag}(\lambda_{1,1}, \dots, \lambda_{n,1})$ where $\lambda_{r_1+1,1} = \dots = \lambda_{n_1,1} = 1$, $\lambda_{n_1+1,1} = \dots = \lambda_{n,1} = 0$. The local (at a_1) change of variables

$$X \mapsto \text{diag}(1, \dots, 1, (t - a_1)^{-1}, \dots, (t - a_1)^{-1})X$$

($n - n_1$ times $(t - a_1)^{-1}$) brings the system to the form

$$\dot{X} = [A_1^*/(t - a_1) + O(1)]X$$

where the matrix A_1^*

- 1) is upper-triangular;
- 2) has no non-zero integer differences between its eigenvalues;
- 3) has an eigenvalue 1 of multiplicity $m_{1,1}$ occupying the last $m_{1,1}$ positions on its diagonal;
- 4) its right lower $m_{1,1} \times m_{1,1}$ -block equals $\begin{pmatrix} I & \Delta \\ 0 & I \end{pmatrix}$; here Δ is the restriction of the matrix

B to the last $n - n_1$ columns intersected with the rows with indices $r_1 + 1, \dots, n_1$.

Hence, the eigenvalues of A_1^* are non-resonant and the monodromy operator M_1^1 is conjugate to $\exp(2\pi i A_1^*)$, see Lemma 32.

Hence, A_1^* and M_1^1 are diagonalizable if and only if $\Delta = 0$. This proves the lemma.

6.3 Proof of Lemma 68

1^0 . Multiply the matrices B_j by $c \in \mathbf{C}^*$ so that the new matrices B_j have strongly generic eigenvalues (the lemma is true or not simultaneously for the old and for the new matrices). It suffices to prove the second equality which would imply that both inequalities $v \geq n - r_1$ and $w \geq v - (n_1 - r_1)$ are equalities. The equality is true exactly if conditions (15) and (16) together are linearly independent. We consider them as a system of linear equations with unknown variables the entries of the vectors $U_j \in \mathbf{C}^{n_1}$ where $T_j = B_j U_j$. Their linear dependence is equivalent to the statement:

there exist vector-rows $V, W \in \mathbf{C}^{n_1}$, $(V, W) \neq (0, 0)$, such that

$$(V + \alpha_j W)B_j = 0 \text{ for } j = 2, \dots, p+1 \text{ and } WB_1 = W \quad (17)$$

Indeed, if (15) and (16) together are not linearly independent, then some non-trivial linear combination of theirs is of the form $0 = 0$. This linear combination is of the form

$$V(\sum_{j=2}^{p+1} B_j U_j) + W(\sum_{j=2}^{p+1} \alpha_j B_j U_j) = \sum_{j=2}^{p+1} (V + \alpha_j W)B_j U_j = 0$$

Its left hand-side must be identically 0 in the entries of U_j , i.e. $(V + \alpha_j W)B_j = 0$ for $j = 2, \dots, p+1$. The condition $WB_1 = W$ follows from $\kappa = r_1 + 1, \dots, n_1$, see (16); recall that B_1 is diagonal and that its last eigenvalue equal to 1 occupies the positions with indices $r_1 + 1, \dots, n_1$, therefore W is left eigenvector of B_1 corresponding to the eigenvalue 1.

2⁰. Consider the fuchsian system

$$\dot{X} = A(t)X, \quad A(t) = \left(\sum_{j=1}^{p+1} \begin{pmatrix} B_j & 0 \\ 0 & 0 \end{pmatrix} / (t - a_j) \right) X \quad (18)$$

of dimension $n_1 + 1$. Perform the change $X \mapsto R(t)X$, $R(t) = \begin{pmatrix} I & 0 \\ V + W/(t - a_1) & 1 \end{pmatrix}$. The matrix $A(t)$ changes to $-R^{-1}\dot{R} + R^{-1}A(t)R$. One can check directly that $A(t)$ does not change under the above change of variables (i.e. $-\dot{R} + A(t)R = RA(t)$) if and only if conditions (17) hold (after the change the system is fuchsian at a_j for $j > 1$ and the residuum equals $\begin{pmatrix} B_j & 0 \\ -(V + W/(a_j - a_1))B_j & 0 \end{pmatrix}$; its polar part at a_1 equals $\begin{pmatrix} 0 & 0 \\ -WB_1 + W & 0 \end{pmatrix} / (t - a_1)^2 + \begin{pmatrix} B_1 & 0 \\ -VB_1 + \sum_{j=2}^{p+1} WB_j/(a_j - a_1) & 0 \end{pmatrix} / (t - a_1)$).

3⁰. The solution to system (18) with initial data $X|_{t=a_0} = I$ changes from X to $R(t)X$ and this must be again a solution to system (18) (because the system does not change). Hence, $R(t)X = XD$ for some $D \in GL(n_1 + 1, \mathbf{C})$. The solution X is block-diagonal (with blocks $n_1 \times n_1$ and 1×1) for all values of t due to the block-diagonal form of the system and, hence, the one of the monodromy group as well.

The first n_1 coordinates of the last column of the matrix $R(t)X$ are identically zero and its restriction to H (the left upper $n_1 \times n_1$ -block) are identically equal to the ones of X . This together with the linear independence of the columns of $X|_H$ implies the form of the matrix D :

$$D = \begin{pmatrix} I & 0 \\ C & g \end{pmatrix}.$$

The conditions $X|_{t=a_0} = I$ and $R(t)X = XD$ imply $g = 1$.

4⁰. The analytic continuations of $R(t)X$ and XD coincide, therefore for every monodromy operator M'_j of the system one must have $R(t)XM'_j = XM'_jD$. But one has $R(t)XM'_j = XDM'_j$, i.e. $[M'_j, D] = 0$ for every monodromy operator.

The monodromy operators are block-diagonal: $M'_j = \begin{pmatrix} M''_j & 0 \\ 0 & 1 \end{pmatrix}$ and the group $\mathcal{G} \subset GL(n_1, \mathbf{C})$ generated by the operators M''_j is irreducible (this follows from the strong genericity of the eigenvalues of the matrices B_j).

The condition $[M'_j, D] = 0$ implies $(M''_j - I)C = 0$ for all j . This together with the irreducibility of the group \mathcal{G} yields $C = 0$. But then $R(t) = I$, i.e. $V = W = 0$ which proves the lemma.

6.4 Proof of Lemma 69

The space Θ of matrices Y_j satisfying 1) is of dimension $(r_2 + \dots + r_{p+1})(n - n_1) = (n + n_1 - r_1)(n - n_1)$ (for $j \geq 2$ the dimension of the image of τ_j is $(n - n_1)r_j$). Its subspace Φ defined by 2) is of codimension $(n - n_1)n_1$ in Θ , hence, of dimension $(n - r_1)(n - n_1)$. This follows from $\dim \tilde{\mathbf{C}}^v = n - r_1$, see Lemma 68, because one has $\Phi = \tilde{\mathbf{C}}^v \times \dots \times \tilde{\mathbf{C}}^v$ ($n - n_1$ times).

The subspace Ξ of Φ defined by condition 3) is of codimension $(n_1 - r_1)(n - n_1)$ in Φ (see Lemmas 64 and 68 – $\dim \tilde{\mathbf{C}}^w = n - n_1$), i.e. of dimension $(n - n_1)^2$.

This dimension is $n - n_1$ times the dimension of the space $\tilde{\mathbf{C}}^w$ of vector-columns Y_j of length n_1 (instead of $n_1 \times (n - n_1)$ -matrices) which satisfy 1) – 3) of the conditions of the lemma. Indeed, one has $\Xi = \tilde{\mathbf{C}}^w \times \dots \times \tilde{\mathbf{C}}^w$ ($n - n_1$ times).

By Lemma 68, $\dim \tilde{\mathbf{C}}^w = n - n_1$, i.e. one can choose exactly $n - n_1$ $(p + 1)$ -tuples of vector-columns satisfying conditions 1) – 3) of the lemma which are linearly independent. The exactitude implies that they are a basis of the space $\tilde{\mathbf{C}}^w$. Hence, the choice of matrices Y_j satisfying 1) – 4) is also possible.

The lemma is proved.

6.5 Proof of Lemma 70

1⁰. Prove 1). A matrix Z commuting with A_1^1 must be of the form $Z = \begin{pmatrix} Z' & 0 \\ 0 & Z'' \end{pmatrix}$, Z'' being $(n - n_1) \times (n - n_1)$.

One must have $Z' = \alpha I$, $\alpha \in \mathbf{C}$ due to Schur's lemma because the $(p + 1)$ -tuple of matrices B_j is irreducible, one has $[A_j^1, Z] = 0$ for all j and, hence, $[B_j, Z'] = 0$.

Hence, for all j one has $\alpha B_j Y_j = B_j Y_j Z''$. The linear independence of the columns of the $(p + 1)$ -tuple of matrices $B_j Y_j$ implies $Z'' = \alpha I$. Part 1) of the lemma is proved.

2⁰. Prove 2). Let $X|_{t=a_0} = I$, $a_0 \neq a_j$, $j = 1, \dots, p + 1$. One can conjugate the monodromy operators (defined for these initial data) to the same form as the one of the matrices-residua: $M_j^1 = \begin{pmatrix} N_j & N_j W_j \\ 0 & I \end{pmatrix}$, N_j being $n_1 \times n_1$, with $W_1 = 0$. If it were known that the columns of the $(p + 1)$ -tuple of matrices $N_j W_j$ are independent, then part 2) of the lemma could be proved like part 1). So suppose that this is not the case.

A conjugation with a matrix $\tilde{D} = \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}$, $D \in GL(n - n_1, \mathbf{C})$, brings the matrices M_j^1 to the form $M_j^1 = \begin{pmatrix} N_j & N_j W_j' & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where I is $(n - n_1 - 1) \times (n - n_1 - 1)$. This conjugation is tantamount to the change of the basis of the solution space: $X \mapsto X \tilde{D}$.

3⁰. Perform the change of the dependent variables $\eta : X \mapsto \begin{pmatrix} I & 0 \\ 0 & D^{-1} \end{pmatrix} X$. This changes system (13) but preserves its block upper-triangular form, the form of its monodromy operators and the size of the blocks B_j .

Hence, after the change for any value of t the solution X is of the form $X = \begin{pmatrix} X' & X'' \\ 0 & I \end{pmatrix}$ because the derivative of any entry of the last $n - n_1$ rows is 0 (recall that the last $n - n_1$ eigenvalues of the matrices A_j^1 before and after the change η are 0). Moreover, one has $X|_{t=a_0} = I$.

4⁰. The form of the monodromy operators implies that each entry of the last column X^n of X is a meromorphic (i.e. univalued) function on \mathbf{CP}^1 . Moreover, the last $n - n_1$ entries of X^n equal identically 0, ..., 0, 1.

Hence, there exists a change of variables $X \mapsto V(t)X$ (with $V(t) = \begin{pmatrix} I & \tilde{V}(t) \\ 0 & 1 \end{pmatrix}$, the matrix-function \tilde{V} being meromorphic on \mathbf{CP}^1 , its last $n - n_1 - 1$ entries being identically 0) after which the new matrix-solution VX is of the form $\begin{pmatrix} X' & X''' & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (I being $(n - n_1 - 1) \times (n - n_1 - 1)$).

Show that system (13) becomes after this change fuchsian again and block-diagonal.

5⁰. Indeed, under the change $X \mapsto V(t)X$ the linear system $\dot{X} = A(t)X$ undergoes the gauge

transformation $A(t) \rightarrow C(t) = -V^{-1}(t)\dot{V}(t) + V^{-1}(t)A(t)V(t)$. Hence, the left $(n-1)$ columns of the matrix $A(t)$ from system (13) do not change at all (we use the fact that the last row of $A(t)$ equals $(0, \dots, 0, 0)$). The last column of the new matrix $A(t) = (\dot{V}X)(VX)^{-1}$ is identically 0, see the form of VX . Hence, the poles of $C(t)$ are of first order and its matrices-residua are of

the form $A_j^2 = \begin{pmatrix} B_j & G_j & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, G_j being $n_1 \times (n - n_1 - 1)$.

6⁰. We show that the matrix V is constant, see 7⁰. This implies that the $(p+1)$ -tuple of matrices-residua A_j^2 is conjugate to the $(p+1)$ -tuple of matrices-residua A_j^1 and, hence, the columns of the $(p+1)$ -tuple of matrices $B_j Y_j$ are dependent – a contradiction. This contradiction proves part 2) the lemma.

7⁰. The change $X \mapsto V(t)X$ from 5⁰ preserves up to conjugacy the matrices-residua A_j^1 for $j > 1$. Represent system (13) and the matrix V in the neighbourhood of a_j by their Laurent series:

$$\dot{X} = [A_j^1/(t - a_j) + O(1)]X, \quad V = V^*/(t - a_j)^k + o(1/(t - a_j)^k), \quad k \geq 0.$$

One has (see 5⁰) $C(t) = (A_j^1)'/(t - a_j) + O(1)$ in the neighbourhood of a_j where $(A_j^1)' \in gl(n, \mathbf{C})$ is conjugate to A_j^1 . The equation $VC = -\dot{V} + AV$ implies $V^*(A_j^1)' = kV^* + A_j^1 V^*$ (these are the coefficients before $1/(t - a_j)^{k+1}$).

If $k > 0$, this equation yields $V^* = 0$. Indeed, the eigenvalues of the linear operator $(.) \mapsto -(.)'(A_j^1)' + k(.) + A_j^1(.)$ acting on $gl(n, \mathbf{C})$ equal $\lambda' = -\lambda_{\nu,j} + k + \lambda_{\mu,j}$. The absence of non-zero integer differences between the eigenvalues of A_j for $j > 1$ implies that $\lambda' \neq 0$, hence, $V^* = 0$, i.e. V has no pole at a_j for $j > 1$.

The form of the last column of the solution X at a_1 and the one of VX imply that $V(a_1) = I$, i.e. V has no pole at a_1 either, hence, no poles on \mathbf{CP}^1 , i.e. V is constant.

The lemma is proved.

6.6 Proof of Lemma 71

Apply the basic technical tool in the multiplicative version, see Subsection 2.2. To prove the lemma it suffices to choose for each j a matrix N_j which is a suitable polynomial of M_j^1 . The $(p+1)$ -tuple of matrices M_j is with trivial centralizer, but can be reducible. Choose N_j such that for $\varepsilon \neq 0$ the eigenvalues of the matrices M_j to be generic. Hence, the $(p+1)$ -tuple of matrices M_j will be irreducible for $\varepsilon \neq 0$.

The lemma is proved.

7 Proof of the necessity in Theorem 58

7.1 Proof of the theorem itself

1⁰. In this section we consider system (1) with generic but not strongly generic eigenvalues, with diagonalizable matrices A_j whose PMV Λ^n is simple and good. Without loss of generality we assume that for $j = 2, \dots, p+1$ one of the eigenvalues of greatest multiplicity of A_j is 0 and for $j = 1$ one of them equals 1 (the last condition is obtained by multiplying the residua by $c \in \mathbf{C}^*$). Hence, the corresponding eigenvalues $\sigma_{k,j}$ of the matrices M_j equal 1, i.e. they satisfy at least one non-genericity relation (denoted by (γ^0)). None of the other eigenvalues $\lambda_{k,j}$ is integer.

2⁰. We assume that for all j the eigenvalues of A_j are non-resonant. We assume also that

A) either (γ^0) is the only non-genericity relation that the eigenvalues $\sigma_{k,j}$ satisfy or the greatest common divisor l of the multiplicities of the non-integer eigenvalues of all matrices A_j is > 1 ; if $l = 1$, then it is possible to choose the eigenvalues $\lambda_{k,j}$ so that the eigenvalues $\sigma_{k,j}$ satisfy only the non-genericity relation (γ^0) and no other. If $l > 1$, then one can divide by l the multiplicities of the eigenvalues $\sigma_{k,j}$ which are not 1 – their product (which is a priori a root of unity of order l , see (5)) might turn out to be a non-primitive such root. This could give rise to another non-genericity relation (γ^1) . In this case one can choose the eigenvalues $\lambda_{k,j}$ so that every non-genericity relation satisfied by the eigenvalues $\sigma_{k,j}$ should be a linear combination of (γ^0) and (γ^1) ;

B) neither $n = 1$, nor the PMV Λ^n satisfies condition (ω_n) (in which cases there is nothing to prove).

3⁰. Assumption B) above implies that the $(p+1)$ -tuple of matrices M_j must be reducible – part 2) of Proposition 9 does not hold (recall that 1 is eigenvalue of greatest multiplicity for all j ; hence, $\text{rk}(M_j - I) = r_j$; if one sets $b_j = 1$, then the necessary condition for existence of irreducible $(p+1)$ -tuples coincides with condition (ω_n) which does not hold).

4⁰.

Lemma 72 *The monodromy group of a fuchsian system with generic non-resonant eigenvalues of the matrices-residua is with trivial centralizer. In particular, the monodromy group of system (1) with eigenvalues defined as above is with trivial centralizer.*

All lemmas from this subsection are proved in the next ones.

Lemma 73 *The monodromy group of system (1) with eigenvalues defined as above can be conjugated to the form $\begin{pmatrix} \Phi & * \\ 0 & I \end{pmatrix}$ where Φ is $n_1 \times n_1$.*

Remark 74 *Notice that the subrepresentation Φ can be reducible.*

Lemma 75 *The centralizer $Z(\Phi)$ of the subrepresentation Φ is trivial.*

5⁰. The subrepresentation Φ being of dimension $n_1 < n$, one can use induction on n to prove the necessity. For $n = 1$ and 2 the necessity is evident. The PMV of the matrices M'_j defining Φ equals Λ^{n_1} . It follows from Lemma 75 that for generic eigenvalues close to the ones of the matrices M'_j defining Φ there exist irreducible $(p+1)$ -tuples of diagonalizable matrices $\tilde{M}'_j \in GL(n_1, \mathbf{C})$ with PMV Λ^{n_1} and satisfying (3) (this can be proved by complete analogy with Lemma 71, by using the basic technical tool in the multiplicative version; recall that the triviality of the centralizer was essential in the proof of Lemma 71 and was assured by Lemma 70).

Hence, if Λ^n is good, then Λ^{n_1} is good. The necessity of (β_n) was proved in Proposition 6 and condition (ω_n) does not hold by assumption. Finally, the PMV Λ^{n_s} is the same for Λ^n and for Λ^{n_1} (this follows from the definition of the PMVs Λ^{n_ν} before Remarks 56 – the PMV Λ^{n_s} is the last of this chain of PMVs). If Λ^{n_1} is good, then either Λ^{n_s} satisfies condition (ω_{n_s}) or one has $n_s = 1$. Hence, if the PMV Λ^n is good, then it satisfies the conditions of Theorem 58, i.e. they are necessary.

The necessity holds in both versions (additive and multiplicative), see Corollary 34.

The necessity is proved.

7.2 Proof of Lemma 72

1⁰. Suppose the lemma not to be true. Then the centralizer either contains a diagonalizable matrix D with at least two distinct eigenvalues or it contains a nilpotent matrix $N \neq 0$. (Indeed, let A and S be respectively a Jordan matrix and its semisimple part. If $[X, A] = 0$, then $[X, S] = 0$.) In the first case we can assume that D has exactly two eigenvalues which can be achieved by considering instead of D some suitable polynomial of it. In the second case without restriction one can assume that $N^2 = 0$ (by considering N^k instead of N for some $k \in \mathbf{N}$).

2⁰. In the first case one conjugates M_j and D to the form $M_j = \begin{pmatrix} M_j^1 & 0 \\ 0 & M_j^2 \end{pmatrix}$, $D = \begin{pmatrix} \alpha I & 0 \\ 0 & \beta I \end{pmatrix}$, $\alpha \neq \beta$. The form of M_j follows from $[M_j, D] = 0$. Apply Proposition 29 twice – once to each of the subspaces defined by the $(p+1)$ -tuples of blocks M_j^1 and M_j^2 . One sees that the sums λ^1, λ^2 of the eigenvalues $\lambda_{k,j}$ corresponding to M_j^1, M_j^2 must be ≤ 0 . On the other hand, there holds $\lambda^1 + \lambda^2 = 0$, see (5), hence, $\lambda^1 = \lambda^2 = 0$. This non-genericity relation contradicts the genericity of the eigenvalues. We used the fact that the eigenvalues are non-resonant – knowing the eigenvalues $\sigma_{k,j}$ of the blocks M_j^i , we know the corresponding eigenvalues $\lambda_{k,j}$ as well (the absence of non-zero integer differences (for fixed j) between the eigenvalues $\lambda_{k,j}$ implies that to equal eigenvalues of M_j there correspond equal eigenvalues of A_j).

3⁰. In the second case the matrices M_j and the matrix N can be conjugated respectively to the form

$$M_j = \begin{pmatrix} M_j' & * & * \\ 0 & M_j'' & * \\ 0 & 0 & M_j' \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where I is $w \times w$, $w \leq n/2$; if $w = n/2$, then the blocks of the middle columns and of the middle rows are absent. The form of M_j follows from $[M_j, N] = 0$.

4⁰. By Proposition 29, the sum λ' of the eigenvalues $\lambda_{k,j}$ corresponding to the block $\begin{pmatrix} M_j' & * \\ 0 & M_j'' \end{pmatrix}$ must be non-positive and the sum λ'' of the eigenvalues $\lambda_{k,j}$ corresponding to the block $\begin{pmatrix} M_j'' & * \\ 0 & M_j' \end{pmatrix}$ must be non-negative (because there holds (5) and the sum of the eigenvalues $\lambda_{k,j}$ corresponding to the upper blocks M_j' is ≤ 0 by Proposition 29). One must have $\lambda' = \lambda''$ (use like in 2⁰ the fact that the eigenvalues are non-resonant). It follows from (5) that $\lambda' = \lambda'' = 0$. This non-genericity condition contradicts the genericity of the eigenvalues $\lambda_{k,j}$.

The lemma is proved.

7.3 Proof of Lemma 73

1⁰. The monodromy group can be conjugated to a block upper-triangular form. The diagonal blocks define either irreducible or one-dimensional representations. The eigenvalues of each diagonal block 1×1 satisfy the non-genericity relation (γ^0).

2⁰. Recall that the integer l was defined in 2⁰ of Subsection 7.1. The block in the right lower corner must be of size 1. Indeed, if $l = 1$, then by Proposition 29 the left upper block cannot be

of size 1 (because the corresponding sum of eigenvalues $\lambda_{k,j}$ equals $1 > 0$). Hence, it must be the only block of size > 1 and the matrices M_j look like this: $M_j = \begin{pmatrix} M'_j & L_j \\ 0 & I \end{pmatrix}$.

The block M' must be of size $\leq n_1$. Indeed, if its size is $> n_1$ (i.e. this is the only diagonal block of size > 1), then the columns of the $(p+1)$ -tuples of matrices L_j are not linearly independent (this is proved by complete analogy with the proof of 4) of Lemma 69).

This proves the lemma in the case $l = 1$.

3⁰. Let $l > 1$. In the absence of second non-genericity relation (γ^1) (it was defined in 2⁰ of Subsection 7.1) the proof is finished like in 1⁰ – 2⁰. So suppose that (γ^1) holds. The diagonal blocks can be of two types. The first are of size 1, the eigenvalues satisfying the non-genericity relation (γ^0).

Describe the second type of diagonal blocks. Their sizes are > 1 and can be different. Define the *unitary set* of eigenvalues: for each j divide by l the multiplicities of all eigenvalues $\sigma_{k,j}$ of the ones that are $\neq 1$. A block F of the second type contains h times the unitary set, $1 \leq h \leq l$, and a certain number of eigenvalues equal to 1. (To different matrices M_j there correspond, in general, different numbers of eigenvalues from the unitary set; therefore one must, in general, add some number of eigenvalues 1 for some values of j to make the number of eigenvalues of the restrictions of the matrices M_j to F equal; one then could eventually add one and the same number of eigenvalues equal to 1 to all matrices $M_j|_F$.)

The eigenvalues of each block of the second type satisfy a corollary of the non-genericity relations (γ^1) and (γ^0).

4⁰. Denote by $\kappa(F)$ the ratio "number of eigenvalues $\sigma_{k,j}$ equal to 1" / "number of eigenvalues $\sigma_{k,j}$ not equal to 1" (eigenvalues of the restriction of the monodromy group to F), and by κ_0 the same ratio computed for the entire matrices M_j (in both ratios one takes into account the eigenvalues of all matrices M_j). Then one must have $\kappa(F) < \kappa_0$.

Indeed, one cannot have $\kappa(F) \geq \kappa_0$ because Λ^n does not satisfy condition (ω_n), hence, the restriction of the monodromy group to F wouldn't satisfy this condition either. In the presence of the non-genericity relation (γ^0) this implies a contradiction with Proposition 9 (like in 3⁰ of Subsection 7.1).

But then the sum $\tilde{\lambda}$ of the eigenvalues $\lambda_{k,j}$ corresponding to the eigenvalues $\sigma_{k,j}$ from F will be negative. If the block F is to be in the right lower corner, then the sum $\tilde{\lambda}$ must be positive (Proposition 29 and (5) – the sum of the eigenvalues of the union of all other diagonal blocks must be ≤ 0 and it cannot be 0 because the eigenvalues $\lambda_{k,j}$ are generic). Hence, the right lower block is of size 1.

5⁰. Denote by Π the left upper $(n-1) \times (n-1)$ -block. Conjugate it to make all non-zero rows of the restriction of the $(p+1)$ -tuple \tilde{M} of matrices $M_j - I$ to Π linearly independent. After the conjugation some of the rows of the restriction of \tilde{M} to Π might be 0. In this case conjugate the matrices M_j by one and the same permutation matrix which places the zero rows of $M_j - I$ in the last (say, m) positions (recall that the last row of $M_j - I$ is 0, see 4⁰, so $m \geq 1$). Notice that if the restriction to Π of a row of $M_j - I$ is zero, then its last (i.e. n -th) position is 0 as well, otherwise M_j is not diagonalizable.

6⁰. Show that $m \geq n - n_1$ (and this will be the end of the proof of the lemma). One has $M_j = \begin{pmatrix} G_j & R_j \\ 0 & I \end{pmatrix}$, $I \in GL(m, \mathbf{C})$.

Denote by \tilde{G} the representation defined by the matrices G_j . We regard the columns of the $(p+1)$ -tuple of matrices R_j as elements of the space $\mathbf{C}'''(\tilde{G})$ defined as follows.

Each column of the $(p+1)$ -tuple of matrices R_j belongs to a linear space $\mathbf{C}'(\tilde{G})$ of dimension

$\theta = r_1 + \dots + r_{p+1}$ which is the sum of the dimensions of the images of the linear operators $(.) \mapsto (G_j - I)(.)$ acting on \mathbf{C}^{n-m} (every column of R_j belongs to the image of this operator, otherwise M_j will not be diagonalizable). Equality (3) is equivalent to $n - m$ linear equations which the entries of the column must satisfy (for the block R this equality implies $\sum_{j=1}^{p+1} G_1 \dots G_{j-1} R_j = 0$; we prove in 7⁰ that these $n - m$ linear equations are linearly independent). Hence, this equality defines a subspace $\mathbf{C}''(\tilde{G})$ of $\mathbf{C}'(\tilde{G})$ of dimension $\theta - (n - m)$.

One then factorizes $\mathbf{C}''(\tilde{G})$ by the space of $(p + 1)$ -tuples of blocks $(G_j - I)V$, $V \in \mathbf{C}^{n-m}$. These blocks are obtained as R -blocks when the $(p + 1)$ -tuple of matrices $\begin{pmatrix} G_j & 0 \\ 0 & 1 \end{pmatrix}$ is conjugated by the matrix $V^* = \begin{pmatrix} I & V \\ 0 & 1 \end{pmatrix}$. This factorization gives the space $\mathbf{C}'''(\tilde{G})$.

7⁰. The space $\mathbf{C}''(\tilde{G})$ is of codimension $n - m$ in $\mathbf{C}'(\tilde{G})$.

One has to show that the $n - m$ linear relations defining $\mathbf{C}''(\tilde{G})$ are linearly independent. If they are not, then the images of all linear operators $(.) \mapsto (G_j - I)(.)$ (acting on \mathbf{C}^{n-m}) must be contained in a proper subspace of \mathbf{C}^{n-m} (say, the one defined by the first $n - m - 1$ vectors of its standard basis). This means that all entries of the last rows of the matrices $G_j - I$ are 0. The matrices M_j being diagonalizable, this implies that the entire $(n - m)$ -th rows of $M_j - I$ are 0. This contradicts the condition the first $n - m$ rows of the restriction to Π of the $(p + 1)$ -tuple of matrices $M_j - I$ to be linearly independent, see 5⁰.

The space $\mathbf{C}'''(\tilde{G})$ is of codimension $n - m$ in $\mathbf{C}''(\tilde{G})$, i.e. of dimension $\theta - 2(n - m)$.

Indeed, each column of V belongs to \mathbf{C}^{n-m} and the intersection \mathcal{I} of the kernels of the operators $(.) \mapsto (G_j - I)(.)$ (acting on \mathbf{C}^{n-m}) is $\{0\}$, otherwise the matrices M_j would have a non-trivial common centralizer. Indeed, if $\mathcal{I} \neq \{0\}$, then after a change of the basis of \mathbf{C}^{n-m} one can assume that a non-zero vector from \mathcal{I} equals ${}^t(1, 0, \dots, 0)$. Hence, the matrices G_j are of the form $\begin{pmatrix} 1 & * \\ 0 & G_j^* \end{pmatrix}$, $G_j^* \in GL(n - m - 1, \mathbf{C})$, and one checks directly that $[M_j, E_{1,n}] = 0$ for $E_{1,n} = \{\delta_{i-1, n-j}\}$.

8⁰. The columns of the $(p + 1)$ -tuple of matrices R_j (regarded as elements of $\mathbf{C}'''(\tilde{G})$) must be linearly independent, otherwise the monodromy group can be conjugated by a matrix $\begin{pmatrix} I & * \\ 0 & P \end{pmatrix}$, $P \in GL(m, \mathbf{C})$, to a block-diagonal form, the right lower block (of size 1) for each j being equal to 1 which means that the monodromy group is a direct sum and, hence, its centralizer is non-trivial – a contradiction with Lemma 72.

This means that $\dim \mathbf{C}'''(\tilde{G}) \geq m$, i.e.

$$\theta - 2(n - m) = r_1 + \dots + r_{p+1} - 2(n - m) \geq m$$

which is equivalent to $m \geq n - n_1$; recall that $n_1 = r_1 + \dots + r_{p+1} - n$. In the case of equality (and only in it) the columns of the $(p + 1)$ -tuple of matrices R_j are a basis of the space $\mathbf{C}'''(\tilde{G})$.

The lemma is proved.

7.4 Proof of Lemma 75

1⁰. If the lemma is not true, then $Z(\Phi)$ either contains a diagonalizable matrix D with exactly two distinct eigenvalues or it contains a nilpotent matrix $N \neq 0$, $N^2 = 0$, see 1⁰ of the proof of Lemma 72.

2⁰. In the first case one can conjugate the monodromy group to the form $\begin{pmatrix} G_j & R_j \\ 0 & I \end{pmatrix}$ with $G_j = \begin{pmatrix} M'_j & 0 \\ 0 & M''_j \end{pmatrix}$ where the sizes of M'_j, M''_j equal the multiplicities of the two eigenvalues of D . One has $D = \begin{pmatrix} \alpha I & 0 \\ 0 & \beta I \end{pmatrix}$, $\alpha \neq \beta$, D and G_j are $n_1 \times n_1$.

3⁰. Denote by $\tilde{M}^{(i)}$ the $(p+1)$ -tuple of matrices $M_j^{(i)}$, $i = 1, 2$. The space $\mathbf{C}'''(\tilde{G})$ was defined in 6⁰ – 7⁰ of the proof of Lemma 73. Set $k' = \dim \mathbf{C}'''(\tilde{M}')$, $k'' = \dim \mathbf{C}'''(\tilde{M}'')$. Hence, $k' + k'' = n - n_1$. Indeed, it follows from the definition of $\mathbf{C}'''(\tilde{G})$ that

$$\mathbf{C}'''(\tilde{G}) = \mathbf{C}'''(\tilde{M}') \oplus \mathbf{C}'''(\tilde{M}'')$$

4⁰. Hence, there exists a conjugation of the monodromy group with a matrix $Q^* = \begin{pmatrix} I & 0 \\ 0 & Q \end{pmatrix}$, $Q \in GL(n - n_1, \mathbf{C})$ after which it is of the form $M_j = \begin{pmatrix} M'_j & 0 & F'_j & 0 \\ 0 & M''_j & 0 & F''_j \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$. Here the columns of the $(p+1)$ -tuples of matrices F', F'' are bases of the spaces $\mathbf{C}'''(\tilde{M}'), \mathbf{C}'''(\tilde{M}'')$. (One first obtains the form $\begin{pmatrix} F'_j & 0 \\ * & F''_j \end{pmatrix}$ of the block R_j by such a conjugation; after this by conjugation with another matrix of the same form as Q^* one makes the block $*$ equal to 0.)

This shows that the monodromy group is a direct sum (one has to perform a self-evident permutation of the blocks and columns to make the matrices block-diagonal). Hence, its centralizer is non-trivial which contradicts Lemma 72.

5⁰. If there exists a matrix $0 \neq N \in Z(\Phi)$, $N^2 = 0$, then one can conjugate N to the form $N = \begin{pmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (or $N = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$) with I being $q \times q$, $q \leq n_1/2$; the second case corresponds to $q = n_1/2$.

6⁰. Hence, $M_j = \begin{pmatrix} M'_j & R_j & T_j & L_j \\ 0 & M''_j & S_j & H_j \\ 0 & 0 & M'_j & P_j \\ 0 & 0 & 0 & I \end{pmatrix}$ where M'_j is $q \times q$ and if $q = n_1/2$, then the blocks

R_j, M''_j, S_j and H_j are absent. In a similar way one brings the blocks $W_j = \begin{pmatrix} L_j \\ H_j \\ P_j \end{pmatrix}$ to the

form $\begin{pmatrix} F'_j & * & * \\ 0 & F''_j & * \\ 0 & 0 & F'_j \end{pmatrix}$ with the same meaning of $M_j^{(i)}$ and $F_j^{(i)}$ as in 3⁰ (such a form of W_j can be achieved by conjugation with a matrix Q^* , see 4⁰).

7⁰. By permuting the rows and columns of M_j (which results from a conjugation) one brings M_j to the form indicated below, with the matrix Z belonging to the centralizer of the monodromy group which again contradicts Lemma 72.

$$M_j = \begin{pmatrix} M'_j & F'_j & R_j & * & T_j & * \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & M''_j & F''_j & S_j & * \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & M'_j & F'_j \\ 0 & 0 & 0 & 0 & 0 & I \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The lemma is proved.

8 On adjacency of nilpotent orbits

Denote by J_1 and J_2 two nilpotent orbits (i.e. conjugacy classes). Let $D_1 \in J_1$, $D_2 \in J_2$. Denote by ρ_i and θ_i the ranks of the matrices $(D_1)^i$ and $(D_2)^i$. It is known that the orbit J_1 belongs to the closure of the orbit J_2 if and only if one has (for all $i \in \mathbf{N}$) $\rho_i \leq \theta_i$ (see [Kr], p. 21).

In this section we prove a more concrete statement, see Theorem 77, from which we deduce Corollary 78. The latter is used in the proof of Theorem 20.

It is evident that if J_1 belongs to the closure of J_2 , then one has for all $i \in \mathbf{N}$ $\rho_i \leq \theta_i$. To prove the implication in the other direction we use the following *operation* (s, l) , defined for $s \geq l$, $s, l \in \mathbf{N}^*$: suppose that the nilpotent orbit J has two Jordan blocks, of sizes s and l , $s \geq l$. We say that the nilpotent orbit J' (of the same size as J) is obtained from J with the help of the operation (s, l) if J' has all Jordan blocks of the same sizes as J except these two which are replaced by two blocks of sizes $s + 1$ and $l - 1$. If $l - 1 = 0$, then only one block of size $s + 1$ replaces the two blocks of sizes s and l .

Proposition 76 *The orbit J lies in the closure of the orbit J' .*

Proof:

1⁰. Assume first that the orbit J is of size $s + l$. Consider the matrix $U(\varepsilon) = \begin{pmatrix} S & \varepsilon R \\ 0 & L \end{pmatrix}$

where S and L are upper-triangular nilpotent Jordan blocks of sizes s and l and $\varepsilon \in \mathbf{C}$. The block R contains a single unit in its lowest row and last column, its other entries are zeros.

2⁰. For $\varepsilon = 0$ one has $U \in J$. For $\varepsilon \neq 0$ the matrix U belongs to J' , i.e. it is conjugate to a nilpotent Jordan matrix with two Jordan blocks, of sizes $l + 1$ and $s - 1$. Indeed, the number and sizes of Jordan blocks of a nilpotent orbit Ω are defined by the ranks of the matrices A^i , $A \in \Omega$. These ranks computed for $U|_{\varepsilon \neq 0}$ and for $V \in J'$ coincide (to be checked directly).

3⁰. It is obvious that the matrix $U|_{\varepsilon=0}$ lies in the closure of the orbit of any of the matrices $U|_{\varepsilon=\varepsilon_0}$, $\varepsilon_0 \neq 0$ (which is J' , i.e. one and the same for all $\varepsilon_0 \neq 0$). This follows from the inclusion of $U(0)$ in the family $U(\varepsilon)$.

4⁰. If the size of the matrices is $> s + l$, then the proposition is proved by analogy (one sets $J = J_0 \oplus J_*$, $J' = J'_0 \oplus J_*$ where J_0, J'_0 are nilpotent orbits of size $l + s$, with blocks of sizes s, l and $s + 1, l - 1$ and J_* is some nilpotent orbit).

The proposition is proved. \square

Theorem 77 *The two orbits J_1 and J_2 can be connected by a chain of intermediate orbits such that each orbit of the chain is obtained from the previous one by some operation (s, l) and, hence, each previous orbit lies in the closure of the next one.*

Proof of Theorem 77:

1⁰. Assume that for the two nilpotent orbits J_1, J_2 there holds $\rho_i \leq \theta_i$ for all i , with strict inequality for at least one i . If each of the orbits contains a Jordan block of size k , then one can decrease the size of the orbits by k by excluding the two equal blocks from consideration. So assume that the two orbits have no such couple of blocks; in particular, that for the blocks of greatest size $H_1 \in J_1$ and $H_2 \in J_2$ one has $h_1 = \text{size}(H_1) < h_2 = \text{size}(H_2)$. Indeed, $h_1 > h_2$ would imply that $\rho_{h_1-1} > \theta_{h_1-1}$.

2⁰. Denote the chain of nilpotent orbits joining J_1 with J_2 by $(J_1, J_3, \dots, J_\nu, J_2)$. Denote by h the size of the second largest Jordan block of J_1 . Then the operation (h_1, h) applied to J_1 preserves the quantities ρ_i for $i \leq h_1 - 1$ and increases ρ_{h_1} by 1 (it changes from 0 to 1).

3⁰. Define J_3 as obtained from J_1 by the operation (h_1, h) . If J_3 coincides with J_2 , then the construction of the chain is finished. If not, then we construct J_4 after J_3 in the same way as J_3 was constructed after J_1 . Namely, if the greatest of the sizes of the Jordan blocks of J_3 , i.e. $h_1 + 1$, equals h_2 , then one can exclude the greatest blocks of J_2 and J_3 from consideration and continue in the same way with orbits of smaller size. This means that the block of size h_2 will be present in all orbits $J_3, J_4, \dots, J_\nu, J_2$. If $h_1 + 1 < h_2$, then one can repeat what was done in 1⁰ – 2⁰ with J_3 on the place of J_1 etc.

4⁰. After finitely many such steps one will have $\rho_i = \theta_i$ for all i , i.e. one obtains the orbit J_2 . Each orbit of the chain is obtained from the previous one by some operation (s, l) . Each previous orbit lies in the closure of the next one, see Proposition 76, so J_1 lies in the closure of J_2 .

The theorem is proved. \square

If the nilpotent orbit J_1 belongs to the closure of the nilpotent orbit J_2 (both of size g), then in general J_1 might have more Jordan blocks than J_2 , i.e. $\text{rk}(J_1) < \text{rk}(J_2)$. In this case we assume that J_2 has $\text{rk}(J_2) - \text{rk}(J_1)$ Jordan blocks of size 0, so that both orbits have the same number of Jordan blocks. When the numbers of Jordan blocks are defined in this way, one can add one and the same number of Jordan blocks of size 0 to J_1 and J_2 . In what follows we assume that the number of Jordan blocks of size 0 is known.

Corollary 78 *Increase by 1 the sizes of the k smallest blocks of J_1 and of the k smallest blocks of J_2 – this defines two nilpotent orbits J'_1, J'_2 , both of size $g + k$. Then J'_1 lies in the closure of J'_2 .*

Remark 79 *It might happen that k is such that one has to increase by 1 only part of the Jordan blocks of a given size. Example: there are 4 Jordan blocks in J_1 , of sizes 2, 2, 1, 1 and $k = 3$. Then the new sizes are 3, 2, 2, 2, i.e. only one of the two blocks of size 2 becomes of size 3.*

Proof:

1⁰. Consider first the case when J_2 is obtained from J_1 by an operation (s, l) .

Denote by m the greatest of the sizes of the k Jordan blocks of J_1 to be increased by 1.

Case 1) One has $m \neq l$ and $m \neq s$. Hence, J'_2 is obtained from J'_1 by an operation

(s, l) if $m < l$,

$(s, l + 1)$ if $l < m < s$,

$(s + 1, l + 1)$ if $s < m$.

Case 2) One has $m = l < s$. Hence, J'_2 is obtained from J'_1 by an operation $(s, l + 1)$.

Case 3) One has $m = l = s$. In this case either $J'_2 = J'_1$ or J'_2 is obtained from J'_1 by an operation $(l + 1, l + 1)$. The first (resp. the second) possibility takes place when not all Jordan

blocks of J_1 of size l have to be chosen as smallest blocks and their sizes increased by 1 (resp. when all of them have to be chosen as such).

Case 4) One has $m = s > l$. Hence, J'_2 is obtained from J'_1 by an operation $(s, l+1)$ or $(s+1, l+1)$. The first (resp. the second) possibility takes place when not all (resp. when all) Jordan blocks of J_1 of size s have to be chosen as smallest blocks and their sizes increased by 1.

Hence, in all these cases J'_1 lies in the closure of J'_2 or coincides with it.

2^0 . In the general case one uses Theorem 77 and applies 1^0 to each couple of consecutive orbits from the chain connecting J_1 and J_2 .

The corollary is proved. \square

9 Proof of Theorem 20

9.1 The basic lemma and its corollaries

It is clear that conditions *i)* and *ii)* from Theorem 17 are necessary for the existence of $(p+1)$ -tuples of matrices with trivial centralizers and with $d = 1$ – the basic technical tool allows one to deform such a $(p+1)$ -tuple \mathcal{A}_0 (resp. \mathcal{M}_0) with a trivial centralizer into a nearby irreducible one \mathcal{A}_1 (resp. \mathcal{M}_1) of matrices from the corresponding diagonal Jordan normal forms and with generic eigenvalues. The condition $d = 1$ implies that the PMV of the eigenvalues of the $(p+1)$ -tuple \mathcal{A}_1 (resp. \mathcal{M}_1) is simple, hence, generic eigenvalues with such PMVs exist. Conditions *i)* and *ii)* from Theorem 17 must hold for \mathcal{A}_1 (resp. for \mathcal{M}_1), hence, they hold for \mathcal{A}_0 (resp. for \mathcal{M}_0) as well, see Remarks 56.

Therefore we prove only the sufficiency of conditions *i)* and *ii)* for the existence of $(p+1)$ -tuples of matrices with trivial centralizers.

Recall that the integers n_i , $i = 0, 1, \dots, s$ were defined before Theorem 17 and that the conditions $n_s > 1$ and (α_n) being a strict inequality are equivalent, see Corollary 61 and Remark 62.

Lemma 80 *If for the Jordan normal forms J_j^n each with a single eigenvalue, with $d = 1$ and with $n_s > 1$ conditions i) and ii) from Theorem 17 hold, then the DSP is weakly solvable for nilpotent matrices A_j^0 with Jordan normal forms J_j^n .*

Remark 81 *The lemma is true also in the case when $d > 1$ and the index of rigidity κ of the $(p+1)$ -tuple of Jordan normal forms is strictly negative. The proof is the same with the exception of 1^0 of it where the four exceptional cases are eliminated due to $d > 1$ and $\kappa < 0$.*

The lemma is proved in the next subsection. It implies the following corollary which finishes the proof of the theorem.

Corollary 82 *1) For any $(p+1)$ -tuple of Jordan normal forms $J_{j;0}^n$ with $d = 1$, with $n_s > 1$ and satisfying conditions i) and ii) from Theorem 17 the DSP is weakly solvable for matrices A_j . In particular, for any generic eigenvalues and such Jordan normal forms it is solvable for matrices A_j .*

2) For any $(p+1)$ -tuple of Jordan normal forms $J_{j;0}^n$ as in 1) and for any generic eigenvalues whose product is 1 the DSP is solvable for matrices M_j .

Proof:

1^0 . Set $A_j^0 = Q_j^{-1} D'_j Q_j$ where A_j^0 are nilpotent and their $(p+1)$ -tuple is with trivial centralizer. Look for matrices A_j of the form $A_j = (I + \varepsilon X_j(\varepsilon))^{-1} Q_j^{-1} (D'_j + \varepsilon D_j) Q_j (I + \varepsilon X_j(\varepsilon))$. Here the matrices D'_j and D_j have the same meaning as D' and Δ_s from Proposition 46.

2⁰. The basic technical tool provides the existence of such matrices A_j for ε small enough. Hence, the matrix A_j is conjugate to εD_j (Proposition 46). One can multiply such a $(p+1)$ -tuple by $1/\varepsilon$ and, hence, in the new $(p+1)$ -tuple the matrix $\varepsilon^{-1}A_j$ will be conjugate to D_j . As the matrices D_j have Jordan normal forms J_j^n and can have any eigenvalues, this proves the corollary for matrices A_j .

3⁰. Prove part 2) using part 1) already proved. One needs to consider the monodromy operators M_j of the fuchsian system (1). The matrices A_j are chosen from the diagonal Jordan normal forms corresponding to $J(M_j)$. The eigenvalues of each matrix A_j form a normalized chain. The condition $d = 1$ implies that the PMV of the eigenvalues of the matrices A_j is simple and one can find strongly generic eigenvalues $\lambda_{k,j}$. The eigenvalues of each matrix A_j form a normalized chain. Hence, for each j one will have $J(M_j) = J_{j;0}^n$ or $J(M_j)$ will be subordinate to $J_{j;0}^n$, see Lemma 52.

The strong genericity of the eigenvalues implies that the monodromy group is irreducible. It follows from Lemma 53 that one can construct an irreducible monodromy group with $J(M_j) = J_{j;0}^n$ for all j .

The corollary is proved. \square

9.2 Proof of Lemma 80

The case $s = 0$.

1⁰. If condition (ω_n) holds, and if $s = 0$ (hence, $n = n_s > 1$), then there exist nice $(p+1)$ -tuples of nilpotent matrices A_j from these nilpotent orbits, see Theorem 13. Indeed, one is never in one of the four exceptional cases cited in Theorem 13 due to $d = 1$. This proves the lemma in the case $s = 0$.

The condition $d = 1$ implies (for any possible value of s , not only for $s = 0$) that the Jordan normal forms $J_j^{n_s}$ never correspond to one of these four exceptional cases. Indeed, in these four cases the PMV of the diagonal Jordan normal forms corresponding to $J_j^{n_s}$ is non-simple. By Lemma 57, the PMV of the diagonal Jordan normal forms corresponding to J_j is non-simple. This implies that $d > 1$ – a contradiction.

The case $s = 1$.

2⁰. If $s = 1$ and if (ω_{n_1}) holds for the Jordan normal forms $J_j^{n_1}$, then there exists a $(p+1)$ -tuple of nilpotent matrices A_j^0 (satisfying (2)) blocked as follows: $A_j^0 = \begin{pmatrix} A'_j & B_j \\ 0 & 0 \end{pmatrix}$ where A'_j are $n_1 \times n_1$, the $(p+1)$ -tuple of conjugacy classes $C(A'_j)$ defines the Jordan normal forms $J_j^{n_1}$ and satisfies condition (ω_{n_1}) . The $(p+1)$ -tuple of matrices A'_j is presumed nice. Show that one can choose the blocks B_j such that

1) for every j the matrix A_j^0 is from the closure of the necessary conjugacy class;

2) the centralizer \mathcal{Z} of the $(p+1)$ -tuple of matrices A_j^0 is trivial.

3⁰. Set

$$\Delta = \{(A'_1 T, \dots, A'_{p+1} T) \mid T \in \mathbf{C}^{n_1}\}.$$

Hence, $\dim \Delta = n_1$. Indeed, the $(p+1)$ -tuple of matrices A'_j being nice implies that the intersection of the kernels of the linear operators $T \mapsto A'_j T$ is $\{0\}$.

4⁰. Every column of the block B_j belongs to a linear space \mathcal{S}_j . It can be described as follows. There exists $m \in \mathbf{N}$ such that all Jordan blocks of size $\geq m$ of A'_j are Jordan blocks of A_j^0 of the same size and all Jordan blocks of size $< m$ have their size increased by 1 in A_j^0 (and, of course, A_j^0 may contain Jordan blocks of size 1 which are not present in A'_j). The matrices A_j^0 and A'_j satisfy the conditions $\text{rk}(A_j^0)^i = \text{rk}(A'_j)^i$ for $i \geq m$.

5⁰. Denote by \mathcal{Q}_j the space $\text{Im}(A'_j)$. Hence, $\dim \mathcal{Q}_j = \text{rk} A'_j$. One can choose as columns of B_j $r_j - \text{rk} A'_j$ vector-columns from $\text{Ker}(A'_j)^{m-1}$ linearly independent modulo \mathcal{Q}_j so that the Jordan normal form of A'_j be the necessary one. Indeed, one can conjugate A'_j so that the block A'_j be in upper-triangular Jordan normal form; the conjugation can be performed by a block-diagonal matrix, with diagonal blocks of sizes $n_1, 1, \dots, 1$.

Conjugate after this the matrix A'_j by a block upper-triangular matrix, the diagonal blocks (of sizes n_1 and $n - n_1$) being equal to I , so that the block B_j contain only zeros in the rows where the units of A'_j are.

After this conjugate A'_j by a block-diagonal matrix, with diagonal blocks of sizes n_1 and $n - n_1$, the first of them being equal to I , so that in the columns of the block B there be exactly one non-zero entry equal to 1 in each of the rows described below (call them *marked*) and in different columns. If A'_j has a Jordan block in the rows with indices $l, l+1, \dots, l+s$ whose size has to be increased by 1 when it is considered as a block of A'_j , then the $(l+s)$ -th row is marked.

(Notice that all these conjugations preserve the size $n_1 \times (n - n_1)$ of the block B_j .)

One checks directly that the Jordan normal form of the matrix A'_j is the necessary one (which is easy to do in the present form of A'_j). The space \mathcal{S}_j is the preimage (before the above conjugations) of the space spanned by the vector-columns having non-zero entries only in the marked rows.

From now on we presume that for every j the $r_j - \text{rk} A'_j$ vector-columns from \mathcal{S}_j (linearly independent modulo the subspace \mathcal{Q}_j) are fixed.

6⁰. Denote by Ω the space of $(p+1)$ -tuples of columns of the blocks B_j modulo the space Δ . Hence, $\dim \Omega \geq r_1 + \dots + r_{p+1} - 2n_1 = n - n_1$. One subtracts n_1 twice because the sum of the $p+1$ columns (of length n_1) must be 0 and to factor out Δ . (We do not need to discuss the question when the inequality is strict and when it is an equality.)

Choose $n - n_1$ $(p+1)$ -tuples of columns of the block B which are linearly independent modulo Δ and whose sum is 0. Check that there holds condition 2) from 2⁰.

Let a matrix $X \in \mathcal{Z}$ equal $\begin{pmatrix} Y & Z \\ T & U \end{pmatrix}$, Y being $n_1 \times n_1$. The commutation relations yield

a) $TA'_j = 0$ for all j , (hence, $TA = 0$ for every matrix from the matrix algebra \mathcal{A} generated by the matrices A'_j ; their $(p+1)$ -tuple being nice, one has $T = 0$)

b) $[Y, A'_j] = 0$ for all j , (hence, $Y = \alpha I$ because the $(p+1)$ -tuple of matrices A'_j is nice)

c) $A'_j Z + B_j U - Y B_j = 0$, i.e. $A'_j Z + B_j U - \alpha B_j = 0$; as the columns of the $(p+1)$ -tuple of blocks B_j are independent modulo the space Δ , i.e. modulo columns of the form $A'_j Z$, one must have $Z = 0$ and $U = \alpha I$. Hence, the centralizer \mathcal{Z} is trivial.

7⁰. Condition 1) from 2⁰ holds, see the construction of the spaces \mathcal{S}_j in 4⁰ and 5⁰. We admit that for some values of j the conjugacy class of A'_j might be not the necessary one but from its closure for the following reason – when one constructs $n - n_1$ $(p+1)$ -tuples of columns of the block B which are linearly independent modulo Δ and whose sum is 0, one does not know whether for each j the vector-columns of B_j of this $(p+1)$ -tuple span the *whole* space $\mathcal{S}_j / \mathcal{Q}_j$.

The case of arbitrary s .

8⁰. For arbitrary s one constructs the $(p+1)$ -tuple of nilpotent matrices A'_j in a similar way: namely, block-decompose any $n \times n$ -matrix, the diagonal blocks being of sizes $n_s, n_{s-1} - n_s, \dots, n_0 - n_1 = n - n_1$. Call *basic minor of size n_{s-k+1}* of a given matrix A (denoted by $A|_{L_k}$) the square submatrix which is the restriction of A to the first n_{s-k+1} rows and n_{s-k+1} columns, $k = 1, \dots, s+1$. Denote by $H_{\mu, \nu}$ the blocks of a matrix from $gl(n, \mathbf{C})$ in this block-decomposition, $\mu, \nu = 1, \dots, s+1$ (we enumerate them in the usual way, from above to below and from left to right).

Denote by \mathcal{Z}_k the centralizer of the $(p+1)$ -tuple of matrices $A_j^0|_{L_k}$. By abuse of language we denote by $J_j^{n_{s-k+1}}$ both the Jordan normal form with a single eigenvalue and the nilpotent orbit defining such a Jordan normal form.

9⁰. The basic minors of the matrices A_j^0 of sizes n_s and n_{s-1} are constructed like in the cases $s=0$ and $s=1$, see 1⁰ – 7⁰. After this the basic minors of sizes n_{s-k} for $k < s+1$ are constructed like the one of size n_{s-1} but one has to take into account the possibility $A_j^0|_{L_k}$ to be from an orbit subordinate to the required one.

Denote by v and w the ranks of $A_j^0|_{L_k}$ and of the required orbit $J_j^{n_{s-k+1}}$ of $A_j^0|_{L_k}$ (i.e. $w = r(J_j^{n_{s-k+1}})$). Hence, $v \leq w$. Define the space $\mathcal{S}_{j,k} \subset \mathbf{C}^{n_{s-k+1}}$ (the analog of \mathcal{S}_j , see 4⁰ – 5⁰). Suppose that $A_j^0|_{L_k}$ is in upper-triangular Jordan normal form (the conjugation of A_j^0 can be carried out by a block-diagonal matrix, with diagonal blocks of sizes $n_{s-k+1}, 1, \dots, 1$).

10⁰. Define as *marked* the rows which are last rows of the b smallest blocks of $A_j^0|_{L_k}$ where

$$b = r(J_j^{n_{s-k}}) - v.$$

If $A_j^0|_{L_k}$ belongs to the closure of the orbit $J_j^{n_{s-k+1}}$ but not to itself, then one might have to choose among the blocks of a given size the size of which ones to be increased by 1, see Remark 79. The space $\mathcal{S}_{j,k}^0$ is defined as spanned by all vector-columns in $\mathbf{C}^{n_{s-k+1}}$ which have units in the rows where $A_j^0|_{L_k}$ has a unit or a marked row. The space $\mathcal{S}_{j,k}$ is the preimage of $\mathcal{S}_{j,k}^0$ under the conjugation of $A_j^0|_{L_k}$.

Hence, $\dim \mathcal{S}_{j,k} = r(J_j^{n_{s-k}})$.

Denote by D_{k+1} the union of the blocks $H_{k,k+1}, H_{k-1,k+1}, \dots, H_{1,k+1}$. Denote by Δ_k the space of $(p+1)$ -tuples of the form

$$(A_1^0|_{L_k} T, \dots, A_{p+1}^0|_{L_k} T) \quad , \quad T \in \mathbf{C}^{n_{s-k+1}}$$

We make two inductive assumptions:

- 1) for all j the orbit of $A_j^0|_{L_k}$ to be either $J_j^{n_{s-k+1}}$ or one subordinate to it;
- 2) the columns of the $(p+1)$ -tuple of matrices $A_j^0|_{L_k}$ to be linearly independent. (This means that $\dim(\Delta_k) = n_{s-k+1}$. The assumption is true for $k=0$ because the algebra \mathcal{A} is nice, and for $k=1$ by the construction of Ω , see 6⁰.)

11⁰. If the columns of $A_j^0|_{D_{k+1}}$ belong to the space $\mathcal{S}_{j,k}$, then the orbit of $A_j^0|_{L_{k+1}}$ will be either $J_j^{n_{s-k}}$ or one subordinate to $J_j^{n_{s-k}}$. This follows from Corollary 78.

Indeed, two cases are possible:

Case 1) One has $n_{s-k} - n_{s-k+1} \geq b$.

If the columns of $A_j^0|_{D_{k+1}}$ belong to the space $\mathcal{S}_{j,k}$, then the orbit of $A_j^0|_{L_{k+1}}$ is either J_0 (obtained from the one of $A_j^0|_{L_k}$ by increasing by 1 the sizes of the $n_{s-k} - n_{s-k+1}$ smallest Jordan blocks of $A_j^0|_{L_k}$) or belongs to the closure of J_0 . Notice that if $n_{s-k} - n_{s-k+1} > b$, then we add $n_{s-k} - n_{s-k+1} - b$ Jordan blocks of size 0 to the orbit of $A_j^0|_{L_k}$.

On the other hand, $J_j^{n_{s-k}}$ is obtained from $J_j^{n_{s-k+1}}$ by increasing by 1 the sizes of the $n_{s-k} - n_{s-k+1}$ smallest blocks (one adds $y := n_{s-k} - n_{s-k+1} - (r(J_j^{n_{s-k}}) - r(J_j^{n_{s-k+1}}))$ Jordan blocks of size 0 to $J_j^{n_{s-k+1}}$ if $y > 0$; recall that one always has $y \geq 0$ – this follows from the definition of the Jordan normal forms $J_j^{n_i}$).

Hence, $J(A_j^0|_{L_{k+1}})$ is either $J_j^{n_{s-k}}$ or is subordinate to it (this follows from assumption 1) from 10⁰ and from Corollary 78).)

Case 2) One has $n_{s-k} - n_{s-k+1} < b$.

If the columns of $A_j^0|_{D_{k+1}}$ belong to the space $\mathcal{S}_{j,k}$, then again the orbit of $A_j^0|_{L_{k+1}}$ belongs to the closure of $J_j^{n_{s-k}}$.

Indeed, increase the sizes of $J_j^{n_{s-k+1}}$ and of $A_j^0|_{L_k}$ not by $n_{s-k} - n_{s-k+1}$ but by b , and increase by 1 the sizes of their b smallest Jordan blocks (one adds the necessary number of Jordan blocks of size 0 to $J_j^{n_{s-k+1}}$).

Denote the matrices thus obtained respectively by L and P . The last $b - n_{s-k} + n_{s-k+1}$ Jordan blocks of L are of size 1. By Corollary 78, P belongs to the closure of the orbit of L (because $A_j^0|_{L_k}$ belongs to the closure of $J_j^{n_{s-k+1}}$). Hence, for all i one has $\text{rk}((P)^i) \leq \text{rk}((L)^i)$. When one reduces the sizes of L and P by deleting their last $b - n_{s-k} + n_{s-k+1}$ columns and rows, the quantities $\text{rk}((L)^i)$ do not change because we delete Jordan blocks of size 1 while the quantities $\text{rk}((P)^i)$ decrease or remain the same (recall that all entries of the last b rows of the matrices P and L are 0).

Hence, for all i one has $\text{rk}((J_j^{n_{s-k}})^i) \geq \text{rk}((A_j^0|_{L_{k+1}})^i)$ and the matrix $A_j^0|_{L_{k+1}}$ belongs to the closure of the nilpotent orbit $J_j^{n_{s-k}}$.

12⁰. The columns of the $(p+1)$ -tuple of blocks $A_j^0|_{D_{k+1}}$ (whose sum is 0) can be chosen linearly independent modulo the space Δ_k .

This is proved by complete analogy with the case $s = 1$, by estimating the dimension of the linear space to which these $(p+1)$ -tuples of columns belong. Namely, the dimension of the space of such $(p+1)$ -tuples of columns is

$$\geq z := r(J_1^{n_{s-k}}) + \dots + r(J_{p+1}^{n_{s-k}}) - \dim(\Delta_k) - n_{s-k+1} = n_{s-k} - n_{s-k+1} \quad (19)$$

because $\dim(\Delta_k) = n_{s-k+1}$ and one has $r(J_1^{n_{s-k}}) + \dots + r(J_{p+1}^{n_{s-k}}) = n_{s-k} + n_{s-k+1}$. Subtracting n_{s-k+1} in (19) corresponds to imposing the n_{s-k+1} conditions the sum of the $(p+1)$ vector-columns to be 0; we do not prove that these conditions are independent, therefore we claim only that the dimension is $\geq z$, not necessarily equal to z .

Hence, the n_{s-k} $(p+1)$ -tuples of columns of the $(p+1)$ -tuple of matrices $A_j^0|_{L_{k+1}}$ are linearly independent. For $k = s+1$ this implies the linear independence of the $(p+1)$ -tuples of columns of the $(p+1)$ -tuple of matrices A_j^0 .

13⁰. The centralizer \mathcal{Z}_{s+1} is trivial.

This is proved like it was done for $s = 1$. Namely, give the decomposition of a matrix from \mathcal{Z}_{s+1} in blocks $H_{\mu,\nu}$.

Consider the blocks $H_{k,1}$ of a matrix $[X, A_j^0]$, $X \in \mathcal{Z}_{s+1}$. One obtains consecutively $X|_{H_{s+1,1}} = \dots = X|_{H_{2,1}} = 0$, $X|_{H_{1,1}} = \alpha I$. Indeed, the $(p+1)$ -tuple of matrices $A_j^0|_{H_{1,1}}$ is nice and the equality $[X, A_j^0]|_{H_{s+1,1}} = 0$ is equivalent to $(XA_j^0)|_{H_{s+1,1}} = 0$ (because $A_j^0|_{H_{s+1,k}} = 0$ for all k, j); the latter implies $X|_{H_{s+1,1}} = 0$ like in a) from 6⁰. Then similarly one deduces that $X|_{H_{s,1}} = 0$ (making use of $X|_{H_{s+1,1}} = 0$) and that $X|_{H_{s-1,1}} = \dots = X|_{H_{2,1}} = 0$. The equality $X|_{H_{1,1}} = \alpha I$ follows from the $(p+1)$ -tuple of matrices $A_j^0|_{H_{1,1}}$ being nice like in b) from 6⁰.

Assume that $\alpha = 0$.

14⁰. Then consider $[X, A_j^0]|_{H_{k,2}}$ for $k = 1, \dots, s+1$. Hence, $X|_{H_{k,2}} = 0$ for $k = 1, \dots, s+1$, otherwise the columns of the $(p+1)$ -tuple of matrices A_j^0 will be linearly dependent. Notice that $[X, A_j^0]|_{H_{k,2}} = 0$ is equivalent to $(A_j^0 X)|_{H_{k,2}} = 0$ because one has $(XA_j^0)|_{H_{k,2}} = 0$ after 13⁰.

Then consider in the same way $[X, A_j^0]|_{H_{k,3}}$, $[X, A_j^0]|_{H_{k,4}}$ etc. One obtains in a similar way that the restrictions of X to $H_{k,3}$, $H_{k,4}$ etc. are 0, i.e. $X = 0$. Without the assumption $\alpha = 0$ this would mean that $X = \alpha I$. Hence, the centralizer is trivial.

15⁰. If for some j the orbit of the matrix A_j^0 is subordinate to the necessary one, i.e. to J_j^n , then one can apply the basic technical tool in the additive version and deform the $(p+1)$ -tuple

of matrices A_j^0 into one satisfying condition 2) from 2^0 and every matrix A_j^0 being from the necessary orbit J_j^n .

The lemma is proved. \square

10 Proof of Theorem 22

In this section we consider only the case of equality in condition (α_n) (i.e. $n_s = 1$). The necessity of conditions *i*) and *ii*) from Theorem 17 for the existence of irreducible $(p+1)$ -tuples of matrices A_j or M_j with generic eigenvalues was proved in Section 7. Their sufficiency for the existence of such $(p+1)$ -tuples for *some* generic eigenvalues was proved in Section 6. There remains to be proved the existence of such $(p+1)$ -tuples (for fixed Jordan normal forms J_j^n) for *all* generic eigenvalues.

10.1 Definitions and notation

Definition 83 *Two $(p+1)$ -tuples of conjugacy classes c_j each with non-resonant eigenvalues and with $\sum_j \text{Tr} c_j = 0$ are said to be similar if they are obtained from one another by adding to equal eigenvalues of a given conjugacy class equal integers; the sum of all added integers (taking into account the multiplicities) is 0.*

Definition 84 *A $(p+1)$ -tuple of conjugacy classes $(c_j$ or $C_j)$ is good (resp. is bad) if the DSP is solvable for matrices $A_j \in c_j$ or for matrices $M_j \in C_j$ (resp. if not). A $(p+1)$ -tuple of Jordan normal forms is good (resp. is bad) if there exists a good $(p+1)$ -tuple of conjugacy classes defining the corresponding Jordan normal forms, with generic eigenvalues (resp. if not). If the $(p+1)$ -tuple of Jordan normal forms is fixed, then the $(p+1)$ -tuple of conjugacy classes is completely defined by the eigenvalues and we say that the eigenvalues are good or bad if the $(p+1)$ -tuple of conjugacy classes is such.*

In the case of matrices A_j and for a fixed $(p+1)$ -tuple of Jordan normal forms denote by \mathbf{C}^s the space of eigenvalues. (If the class c_j has s_j distinct eigenvalues, then $s = s_1 + \dots + s_{p+1}$). Denote by $\mathbf{C}' \simeq \mathbf{C}^{s-1}$ the subspace of \mathbf{C}^s defined by the condition the sum of all eigenvalues (taking the multiplicities into account) to be 0. The PMV of the eigenvalues is simple, otherwise there exist no generic eigenvalues at all.

In accordance with the above definition, we call the points from \mathbf{C}' *good* or *bad* if they define good or bad $(p+1)$ -tuples of conjugacy classes. A point from \mathbf{C}' is called *(strongly) generic* (resp. *non-resonant*) if it defines (strongly) generic (resp. non-resonant) eigenvalues.

Denote by Ω the set of good points of \mathbf{C}' . By Lemma 25 (in which L coincides with \mathbf{C}' ; recall that L is connected in the case of matrices A_j), the set Ω contains a Zariski open dense subset of \mathbf{C}' . The set Ω is constructible and invariant under multiplication by \mathbf{C}^* .

10.2 Proof of Theorem 22 in the multiplicative version

1^0 . If the $(p+1)$ -tuple of Jordan normal forms J_j^n is good, then it is impossible to have the following situation: there exists a bad non-resonant strongly generic point $P \in \mathbf{C}'$ and every point from the set Θ_P consisting of P and of all points in \mathbf{C}' defining $(p+1)$ -tuples of conjugacy classes similar to the ones defined by P is also bad. Indeed, the constructibility of Ω implies that if the above situation takes place, then Ω cannot contain a Zariski open dense subset of the set of all generic eigenvalues of \mathbf{C}' ; hence, Ω must be empty.

2⁰. Hence, every set Θ_P defined like above contains a good strongly generic non-resonant point. Every strongly generic non-resonant point from \mathbf{C}' defines a $(p+1)$ -tuple of conjugacy classes in $GL(n, \mathbf{C})$ via the rule: if c_j are the conjugacy classes in $gl(n, \mathbf{C})$ defined by the point and if $A_j \in c_j$, then C_j are the conjugacy classes of the matrices $\exp(2\pi i A_j)$.

3⁰. All points from Θ_P define one and the same $(p+1)$ -tuple of conjugacy classes $C_j \in GL(n, \mathbf{C})$, with generic eigenvalues. Hence, this $(p+1)$ -tuple of conjugacy classes is good (indeed, the monodromy group of a fuchsian system with matrices-residua $A_j \in c_j$ is irreducible for any choice of the positions of the poles and one has $M_j \in C_j$ with c_j as in 2⁰). On the other hand, for every $(p+1)$ -tuple of conjugacy classes in $GL(n, \mathbf{C})$ with generic eigenvalues one can find a set Θ_P as above which defines this $(p+1)$ -tuple of conjugacy classes.

4⁰. Hence, if a given $(p+1)$ -tuple of Jordan normal forms is good, then for all possible generic eigenvalues there exist $(p+1)$ -tuples of matrices M_j with this $(p+1)$ -tuple of Jordan normal forms. This proves Theorem 22 in the multiplicative version.

10.3 Proof of Theorem 22 in the additive version

1⁰. Consider a good generic point D from \mathbf{C}' such that all eigenvalues of all conjugacy classes c_j are integer (hence, it is not strongly generic). All such points from \mathbf{C}' cannot be bad because the constructibility of Ω would imply that Ω is empty.

Lemma 85 *The monodromy operator of a fuchsian system the eigenvalues of whose matrices-residua define the point D is upper-triangular up to conjugacy.*

The lemma is proved in the next subsection.

The lemma implies that one can choose an initial value of the solution X such that the monodromy group is upper-triangular, with matrices M_j arbitrarily close to I in some matrix norm. Hence, for matrices-residua close to the given ones defined by the point D the monodromy operators M_j will be all close to I . If these matrices-residua are with the same Jordan normal forms as the ones defined by D , then the monodromy group is defined by a point from a set $\tilde{D} \subset \mathbf{C}'$ containing a neighbourhood of D in \mathbf{C}' . These points are also good – to prove it one has to apply the basic technical tool in the additive version.

2⁰. In [L-D] I.A. Lappo-Danilevskii proves the following result:

The monodromy operators of a fuchsian system are expressed as power series of its matrices-residua. These series are convergent if the residua are small enough and for such residua the map "residua" \mapsto "monodromy operators" is a diffeomorphism of a neighbourhood \mathcal{N}_0 of 0 to a neighbourhood \mathcal{N}_1 of I . The initial data are $X|_{t=\infty} = I$ assuming that there is no pole at ∞ .

3⁰. We give another formulation of the above result. Identify the space of $(p+1)$ -tuples of matrices A_j whose sum is 0 with $(gl(n, \mathbf{C}))^p$ (one defines only the first p of them). Denote by \mathcal{S} the unit sphere in $(gl(n, \mathbf{C}))^p$ when regarded as \mathbf{R}^{2pn^2} . Introduce coordinates in $(gl(n, \mathbf{C}))^p$ which are the union of some coordinates on \mathcal{S} and $h \geq 0$. Consider the fuchsian system

$$\dot{X} = \left(\sum_{j=1}^{p+1} h A_j / (t - a_j) \right) X$$

(without a pole at ∞ , with $(A_1, \dots, A_{p+1}) \in \mathcal{S}$) and its solution X satisfying the condition $X(\infty) = I$.

Lemma 86 *For h small enough and $(A_1, \dots, A_{p+1}) \in \mathcal{S}$ one has $M_j = I + h 2\pi i A_j + o(h)$. The estimation is uniform in $(A_1, \dots, A_{p+1}) \in \mathcal{S}$.*

The lemma is proved in Subsection 10.5. It implies that for h small enough the map "residua" \mapsto "monodromy operators" is a diffeomorphism of \mathcal{N}_0 to \mathcal{N}_1 .

4⁰. The monodromy group \mathcal{M} of a fuchsian system (F_1) with conjugacy classes of its matrices-residua corresponding to every strongly generic non-resonant point from \tilde{D} admits a conjugation after which it will belong to \mathcal{N}_1 , see 2⁰.

By 3⁰, there exists a fuchsian system (F_2) with eigenvalues close to 0 with the same monodromy group. Its eigenvalues are shifted w.r.t. the ones of (F_1) by integers (and the $(p+1)$ -tuples of conjugacy classes of the matrices-residua of (F_1) and (F_2) are similar). These integers are opposite to the eigenvalues of the $(p+1)$ -tuple of conjugacy classes defined by the point D .

5⁰. Every strongly generic non-resonant point in \mathbf{C}' which is close to D is good and by the shift of eigenvalues defined in 4⁰ it defines a good strongly generic non-resonant point close to 0. The shift leaves the set of strongly generic non-resonant points invariant. Hence, all strongly generic non-resonant points close to 0 are good.

This means that all generic points close to 0 are good. Indeed, a generic point being close to 0 means that it is strongly generic and non-resonant.

6⁰. By 5⁰, the set Ω contains the intersection of some neighbourhood of $0 \in \mathbf{C}'$ with the set of generic points of \mathbf{C}' . The set Ω being invariant under multiplication by \mathbf{C}^* (it is defined by linear homogeneous inequalities), it must contain the set of all generic points of \mathbf{C}' .

This proves Theorem 22 in the additive version.

10.4 Proof of Lemma 85

1⁰. Suppose that the monodromy group is not triangularizable by conjugation. Then it can be conjugated to a block upper-triangular form, its restriction to at least one diagonal block P of size $m > 1$ being irreducible. Hence, the matrix algebra generated by the restriction of the monodromy matrices M_j to the block P is $gl(m, \mathbf{C})$ (the Burnside theorem) and, hence, there exists a polynomial without a constant term $s(M_1 - I, \dots, M_{p+1} - I)$ in the matrices $M_j - I$ which is a matrix with at least two distinct eigenvalues.

2⁰. All points from \mathbf{C}' sufficiently close to D are good as well (this follows easily from the basic technical tool in the additive version). Hence, the same polynomial s evaluated for M_j corresponding to points close to D is still a matrix with at least two distinct eigenvalues which will be close to two of the eigenvalues of s evaluated at D (denoted by a, b). Suppose that $a \neq 0$. Hence, for all points from \mathbf{C}' close to D (denote their set by D^*) the polynomial s evaluated at them has an eigenvalue λ with $|\lambda| \geq |a|/2$.

3⁰. *There exists a constant $\delta > 0$ such that $\|s\| \geq \delta$ for all points from D^* .* (Notice that this estimation is based only on the presence of distinct eigenvalues; the monodromy group is defined only up to conjugacy and the above estimation is valid for any of the possible definitions of the monodromy group.)

Suppose that such a constant δ does not exist. Then for each $\varepsilon > 0$ there exist points from D^* arbitrarily close to D such that $\|s\| < \varepsilon$. One can choose as matrix norm the sum of the absolute values of all entries of a matrix.

There holds the following lemma (well-known to specialists in numerical methods – the lemma of diagonal domination):

Lemma 87 *If the module of any diagonal entry of a matrix from $gl(n, \mathbf{C})$ is greater than the sum of the modules of the non-diagonal entries of the same row, then the matrix is non-degenerate.*

Hence, if the modules of all entries of a matrix A from $gl(n, \mathbf{C})$ are smaller than $|a|/4n$, then the matrix cannot have an eigenvalue λ with $|\lambda| \geq |a|/2$ (because the matrix $A - \lambda I$ will satisfy

the conditions of the lemma). This implies the existence of δ as above.

4⁰. Fix a generic point $Q \in \Omega$ and consider the points hQ , $h \in [0, 1]$. Denote their set by γ . For $h \neq 0$ they are generic and belong to Ω ; for $h \neq 0$ small enough they are strongly generic and non-resonant. Let the $(p+1)$ -tuple $(A_1^*, \dots, A_{p+1}^*)$ be with eigenvalues defined by Q . Consider the fuchsian system with fixed poles and with $(p+1)$ -tuple of matrices-residua hA_j^* . If one defines its monodromy group by fixing the initial point and initial value of X one and the same for all h , then for h small enough its monodromy operators $M_j(h)$ will be arbitrarily close to I . This follows from the continuous dependence of the solution on the parameter h (when the solution is considered on any simply connected domain not containing a pole of the system) – for $h = 0$ one has $X = \text{const}$, $M_j = I$. Hence, the norm of the polynomial s computed for these monodromy groups will be arbitrarily small when $h \rightarrow 0$.

For $h \neq 0$ small enough the strong genericity of the eigenvalues of the matrices hA_j^* implies that the monodromy groups of the systems are irreducible. They are rigid because (α_n) is an equality. Hence, they are unique up to conjugacy.

5⁰. Denote by $\gamma' \subset \mathbf{C}'$ the segment γ translated so that the point corresponding to $h = 0$ be at D . For h small enough the points from γ' belong to Ω . The translation of γ into γ' means that for every fixed value $h_0 \neq 0$ of h its corresponding $(p+1)$ -tuple of conjugacy classes defined by the point $R = \gamma|_{h=h_0}$ will be replaced by a similar $(p+1)$ -tuple defined by the point $R' = \gamma'|_{h=h_0}$. Hence, up to conjugacy, the monodromy groups of the fuchsian systems with matrices-residua whose eigenvalues are defined by the points R and R' coincide (when h_0 is small enough). Indeed, they correspond to similar $(p+1)$ -tuples of conjugacy classes.

This however is a contradiction with 2⁰ – by choosing h small enough the norm of s computed for R can become arbitrarily small which is impossible to happen for R' , see 3⁰.

The lemma is proved. \square

10.5 Proof of Lemma 86

1⁰. To compute the monodromy operators M_j we fix the contours of integration. They begin at ∞ , go along arcs η_j to some points b_j close to a_j , go around a_j counterclockwise along the circumferences ζ_j passing through b_j and centered at a_j , and then go back to ∞ along η_j . The points b_j and the arcs η_j are chosen such that there is no other pole of the system except a_j on the closed discs Ξ_j (where $\partial\Xi_j = \zeta_j$) and no pole at all on η_j .

2⁰. For each j the value at b_j of the analytic continuation of the solution to the system with initial data $X(\infty) = I$ along the segment η_j equals $I + Rh + o(h)$. This estimation is uniform in $(A_1, \dots, A_{p+1}) \in \mathcal{S}$. Indeed, this follows from the smooth dependence of the solution X on h (there are no singularities of the system on η_j ; for $h = 0$ one has $X \equiv I$).

3⁰. Denote by K_j the operators of local monodromy defined with initial data $X(b_j) = I$ and mapping this value onto the value of the analytic continuation of X along ζ_j . For h small enough one has $K_j = I + h2\pi i A_j + o(h)$.

Indeed, for h small enough and for any $(A_1, \dots, A_{p+1}) \in \mathcal{S}$ no two eigenvalues of any of the matrices A_j differ by a non-zero integer. Hence, the solution to the system in some neighbourhood of a_j can be represented in the form

$$X = (I + P(t, h)) \exp(hA_j \ln(t - a_j)) G_j$$

where $G_j \in GL(n, \mathbf{C})$ and P is a Taylor series in $(t - a_j)$ whose terms are expressed through the entries of the matrices A_j , see [Wa]. One has $P = O(h)$, $P(a_j, h) = 0$. If $X(b_j) = I$, then

$$G_j = \exp(-hA_j \ln(b_j - a_j)) (I + P(b_j, h))^{-1} \quad (20)$$

The matrix-function $(I + P(t, h))$ is holomorphically invertible on $\Xi_j \setminus a_j$ because G_j , X and $\exp(hA_j \ln(t - a_j))$ are such (the latter two are multivalued). As $P(a_j, h) = 0$, it is holomorphically invertible inside Ξ_j , i.e. at a_j as well.

4⁰. The monodromy of the matrix-function $\exp(hA_j \ln(t - a_j))$ around a_j equals $\exp(h2\pi i A_j) = I + h2\pi i A_j + o(h)$. One has

$$K_j = (G_j)^{-1} \exp(h2\pi i A_j) G_j = (G_j)^{-1} (I + h2\pi i A_j + o(h)) G_j$$

where G_j is defined by (20). The factor $\exp(-hA_j \ln(b_j - a_j))$ commutes with $\exp(h2\pi i A_j)$. The factor $(I + P(b_j, h))^{-1}$ equals $I + Qh + o(h)$. Hence,

$$K_j = (I + Qh + o(h))^{-1} (I + h2\pi i A_j + o(h)) (I + Qh + o(h)) = I + h2\pi i A_j + o(h).$$

5⁰. The operator M_j is conjugate to K_j and the conjugation is carried out by the matrix $I + Rh + o(h)$ from 2⁰. Hence, $M_j = I + h2\pi i A_j + o(h)$.

The lemma is proved. \square

11 Proof of Theorem 19

1⁰. Prove the necessity. Denote by J_j^n the Jordan normal forms of the matrices M_j and by $J_j^{n,1}$ their corresponding diagonal Jordan normal forms, see Subsection 4.1. The PMV of the $(p+1)$ -tuple of Jordan normal forms $J_j^{n,1}$ is simple; this follows from $d = 1$ and from the construction of J_1 after J_0 in Subsection 4.1.

2⁰. Apply the basic technical tool in the multiplicative version. Look for matrices \tilde{M}_j of the form

$$\tilde{M}_j(\varepsilon) = (I + \varepsilon X_j(\varepsilon))^{-1} Q_j^{-1} (G_j^0 + \varepsilon G_j^1) Q_j (I + \varepsilon X_j(\varepsilon))$$

where $M_j = Q_j^{-1} G_j^0 Q_j = \tilde{M}_j(0)$, G_j^0 being Jordan matrices and G_j^1 being diagonal matrices. The matrices G_j^0 and G_j^1 are chosen with the properties respectively of G^0 and G^1 from part 2) of Proposition 44. For ε small enough the eigenvalues of the matrices \tilde{M}_j are generic and their Jordan normal forms (when $\varepsilon \neq 0$) are $J_j^{n,1}$, see Proposition 44.

3⁰. Theorem 58 gives the necessary and sufficient conditions for the existence of irreducible $(p+1)$ -tuples of matrices $\tilde{M}_j \in J_j^{n,1}$ with generic eigenvalues. It follows from the theorem that conditions *i*) and *ii*) of Theorem 17 hold for the $(p+1)$ -tuple of Jordan normal forms J_j^n , see Remarks 56. Hence, conditions *i*) and *ii*) from Theorem 17 are necessary for the existence of irreducible $(p+1)$ -tuples of matrices M_j .

This proves the necessity.

4⁰. Prove the sufficiency. If the conditions of Theorem 58 hold for the diagonal Jordan normal forms $J_j^{n,1}$ (recall that their PMV is simple), then there exist irreducible $(p+1)$ -tuples of matrices $A_j^2 \in J_j^{n,1}$ satisfying (2). One can choose their eigenvalues $\lambda_{k,j}$ (presumed to be generic and to form for each j a normalized chain, see Subsection 4.2) such that $\exp(2\pi i \lambda_{k,j}) = \sigma_{k,j}$. Hence, for each j the Jordan normal form of M_j will be either J_j^n or some Jordan normal form subordinate to it, see Lemma 52. By part 2) of Lemma 53, there exist irreducible $(p+1)$ -tuples of matrices $M_j \in J_j^n$, with generic eigenvalues.

The theorem is proved. \square

12 Proofs of Propositions 6 and 9

1⁰. We prove (α_n) in $1^0 - 4^0$ and (β_n) in 5^0 . To find the dimension of the representation defined by the matrices A_j in $(sl(n, \mathbf{C}))^p$ one has first to restrict oneself to the cartesian product $c_1 \times \dots \times c_p$ of the orbits of A_1, \dots, A_p considered as varieties in each of the p copies of $sl(n, \mathbf{C})$.

2⁰. The algebraic variety \mathcal{V} defined in $sl(n, \mathbf{C})^p$ by the orbits of A_1, \dots, A_{p+1} is the projection in $c_1 \times \dots \times c_p$ of the intersection of the two varieties in $c_1 \times \dots \times c_p \times sl(n, \mathbf{C})$ (the matrix A_{p+1} corresponds to the factor $sl(n, \mathbf{C})$): the graph of the mapping

$$(c_1 \times \dots \times c_p) \ni (A_1, \dots, A_p) \mapsto A_{p+1} = -A_1 - \dots - A_p \in sl(n, \mathbf{C})$$

and $c_1 \times \dots \times c_p \times c_{p+1}$. This intersection is transversal which implies the smoothness of the variety \mathcal{V} (this can be proved by analogy with 1) of Theorem 2.2 from [Ko5]). Thus

$$\dim \mathcal{V} = \left(\sum_{j=1}^p \dim c_j \right) - [(n^2 - 1) - \dim c_{p+1}]$$

(here $(n^2 - 1) - \dim c_{p+1} = \text{codim}_{sl(n, \mathbf{C})} c_{p+1}$). Hence, $\dim \mathcal{V} = \sum_{j=1}^{p+1} \dim c_j - n^2 + 1$.

3⁰. In order to obtain the dimension of the representation defined by (A_1, \dots, A_{p+1}) one has to factor out the possibility to conjugate the $(p+1)$ -tuple (A_1, \dots, A_{p+1}) with matrices from $SL(n, \mathbf{C})$. No such non-scalar matrix commutes with all the matrices (A_1, \dots, A_{p+1}) due to the irreducibility of the $(p+1)$ -tuple and to Schur's lemma. Thus the dimension of the representation equals

$$\dim \mathcal{V} - n^2 + 1 = \sum_{j=1}^{p+1} \dim c_j - 2n^2 + 2$$

(where $-n^2 + 1 = -\dim SL(n, \mathbf{C})$). When this dimension is negative, then the representation does not exist. In the case of $(SL(n, \mathbf{C}))^p$ the mapping $(A_1, \dots, A_p) \mapsto A_{p+1} = -A_1 - \dots - A_p$ from **2⁰** has to be replaced by the mapping

$$(M_1, \dots, M_p) \mapsto M_{p+1} = (M_1 \dots M_p)^{-1}.$$

This proves (α_n) .

4⁰. Prove (β_n) . If

$$r_1 + \dots + \hat{r}_j + \dots + r_{p+1} < n,$$

then for suitable eigenvalues λ_i

$$\text{codim} \cap_{i \neq j} \text{Ker}(A_i - \lambda_i I) < n$$

and this non-trivial subspace contradicts the irreducibility assumption.

The proposition is proved. \square

Proof of Proposition 9:

1⁰. The $(p+1)$ -tuples of matrices A_j and $A_j - b_j I$ being simultaneously irreducible we prove the proposition for $b_j = 0$, $j = 1, \dots, p+1$. Consider the maps

$$\begin{aligned} \sigma : (X_1, \dots, X_{p+1}) &\mapsto (A_1 X_1, \dots, A_{p+1} X_{p+1}) && \text{and} \\ \tau : (X_1, \dots, X_{p+1}) &\mapsto (A_1 X_1 + \dots + A_{p+1} X_{p+1}) \quad , \quad X_j \in \mathbf{C}^n \end{aligned}$$

Prove that $\dim(\text{Im } \sigma) \geq 2n$ (which amounts to proving part 1) of the lemma).

Denote by S_1 the subspace of $\mathbf{C}^{(p+1)n}$ where $S_1 = \{(X, \dots, X), X \in \mathbf{C}^n\}$, $\dim S_1 = n$; one has $\tau(S_1) = \{0\}$ (because $A_1 + \dots + A_{p+1} = 0$). The irreducibility of (A_1, \dots, A_{p+1}) implies that

A) the image of τ is the whole space \mathbf{C}^n ; hence, there exists a subspace $S_2 \subset \mathbf{C}^{(p+1)n} \simeq \{(X_1, \dots, X_{p+1})\}$, $\dim S_2 = n$, such that $\tau(S_2) = \mathbf{C}^n$;

B) $S_1 \cap \text{Ker } \sigma = \{0\}$.

Hence,

a) $S_1 \cap S_2 = \{0\}$ (otherwise a vector from S_2 belongs to $\text{Ker } \tau$ which contradicts $\tau(S_2) = \mathbf{C}^n$ and $\dim S_2 = n$), $\dim(S_1 \oplus S_2) = 2n$ and

b) the image of $S_1 \oplus S_2$ under σ is of dimension $2n$. Indeed, the image of S_1 is of dimension n and belongs to $\text{Ker } \tau$ whereas the one of S_2 (also of dimension n) is transversal to $\text{Ker } \tau$. This proves 1).

2^o. Prove 2). If the monodromy group generated by the matrices M_j is irreducible, then so is the algebra generated by the matrices M_j , $j = 1, \dots, p+1$; hence, by the matrices $N_j = (b_{j+1} \dots b_{p+1})M_1 \dots M_{j-1}(M_j - b_j I)$ as well (recall that $b_1 \dots b_{p+1} = 1$).

One has $N_1 + \dots + N_{p+1} = 0$ and $\text{rk } N_j = \text{rk}(M_j - b_j I)$. Hence, part 2) follows from part 1) by setting $A_j = N_j$.

The proposition is proved. \square

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